Robust Aspects of Hedging and Valuation in Incomplete Markets and related Backward SDE Theory

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M.Sc. Klébert, Kentia Tonleu

Präsident der Humboldt-Universität zu Berlin
Prof. Dr. Jan-Hendrik Olbertz

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät
Prof. Dr. Elmar Kulke

Gutachter: 1. Prof. Dr. Dirk Becherer
2. Prof. Dr. Romuald Elie
3. Prof. Dr. Peter Imkeller

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Abstract

This thesis studies backward stochastic differential equations (BSDEs), and robust notions of dynamic good-deal valuation and hedging in incomplete financial markets.

We start by a mathematical theory, concerning the analysis of BSDEs with jumps driven by random measures that can be of infinite activity with time-inhomogeneous compensators. Under concrete conditions that are easy to verify in practical applications, we provide existence, uniqueness and comparison results for bounded solutions for a class of generator functions that are not required to be globally Lipschitz in the jump integrand. To illustrate the ease of applicability of our results, we solve the exponential and power utility maximization problems with additive and multiplicative liability respectively.

The rest of the thesis deals with the more application-oriented problem of robust valuation and hedging in incomplete markets. We are concerned with the no-good-deal approach, which computes good-deal valuation bounds by using as pricing measures only a subset of the risk-neutral measures satisfying a constraint on the Girsanov kernels described by correspondences with economic meaning. Examples of such constraints are given by bounds on instantaneous Sharpe ratios, optimal growth rates, or expected utilities. Throughout we study a notion of good-deal hedging that corresponds to good-deal valuation, and for which hedging strategies arise as minimizers of some dynamic coherent risk measures allowing for optimal risk sharing with the market. Hedging is shown to be at least mean-self-financing in the sense that tracking (hedging) errors satisfy a supermartingale property under suitable a-priori valuation measures. The latter is then interpreted as robustness of good-deal hedging, with respect to the family of valuation measures as generalized scenarios.

We derive constructive results on good-deal valuation and hedging using BSDEs. The results are obtained in a jump framework with unpredictable event-risk, as well as in a Brownian setting with model uncertainty. In the jump framework we use the theory on BSDEs with jumps, and provide examples in (semi-)Markovian models, which are particularly relevant for actuarial applications. In the Brownian setting, we provide new examples for concrete no-good-deal constraints, with closed-form expressions for valuations and hedges given via classical option pricing formulas (Black-Scholes, Margrabe or Heston). Moreover, under Knightian uncertainty (ambiguity) about the objective real-world probability measure which is not taken to be precisely known, we study robustness of hedging taking into account the investor's aversion towards ambiguity. Assuming multiple reference priors as candidates for the (uncertain) real-world measure, a worst-case approach leads to good-deal hedging that is robust with respect to uncertainty in the sense that it is at least mean-self-financing uniformly over all priors. Results are presented for drift uncertainty and volatility uncertainty separately, using classical BSDEs for the former and second-order BSDEs for the latter. Under drift uncertainty, we also show existence of a worst-case prior with respect to which dynamic valuations and hedges can be computed like in the absence of uncertainty. Here the robust approach yields that good-deal hedging is equivalent to risk-minimization with respect to a suitable measure if drift uncertainty is sufficiently large. In the case of volatility uncertainty, we provide an example for put options in an uncertain volatility model of Black-Scholes' type, where explicit solutions for (robust) good-deal valuations and hedges are computable under a worst-case prior.

Zusammenfassung

Diese Arbeit untersucht stochastische Rückwärtsdifferentialgleichungen (BSDEs) und robuste Konzepte von dynamischer Good-Deal-Bewertung und -Hedging in unvollständigen Finanzmärkten.

Wir beginnen mit einer mathematischen Theorie zur Analyse von BSDEs mit Sprüngen, getragen von zufälligen Maßen, die von unendlicher Aktivität mit zeitlich inhomogenem Kompensator sein können. Unter konkreten Bedingungen, die in praktischen Anwendungen leicht zu verifizieren sind, liefern wir Existenz-, Eindeutigkeits- und Vergleichsergebnisse beschränkter Lösungen für eine Klasse von Generatorfunktionen, welche nicht-notwendigerweise global Lipschitz-stetig im Sprungintegranden sein müssen. Wir lösen das Maximierungsproblem für exponentiellen Nutzen bei additiver Verbindlichkeit und für Power-Nutzen bei multiplikativer Verbindlichkeit, um die Anwendbarkeit unserer Resultate zu veranschaulichen.

Der übrige Teil der Arbeit beschäftigt sich mit dem eher anwendungsorientierten Problem der robusten Bewertung und des Hedgings in unvollständigen Märkten. Wir befassen uns mit dem No-Good-Deal-Ansatz, welcher Good-Deal-Grenzen liefert, indem als Bewertungsmaße lediglich eine Teilmenge der risikoneutralen Maße betrachtet werden, die eine Bedingung an den Girsanov-Kern – beschrieben durch Korrespondenzen mit ökonomischer Bedeutung – erfüllen. Beispiele solcher Bedingungen sind Grenzen für instantanen Sharpe-Ratio, optimale Wachstumsrate oder erwarteten Nutzen. Durchweg untersuchen wir ein Konzept des Good-Deal-Hedgings, das Good-Deal-Bewertung entspricht und für welches Hedgingstrategien als Minimierer geeigneter dynamischer kohärenter Risikomaße auftreten, was optimale Risikoteilung mit der Markt erlaubt. Wir zeigen, dass Hedging mindestens im-Mittel-selbstfinanzierend ist. Das heißt, dass Hedgefehler unter geeigneten A-priori-Bewertungsmaßen eine Supermartingaleigenschaft haben. Dies wird als Robustheit des Good-Deal-Hedgings bezüglich der Familie von Bewertungsmaßen, gesehen als verallgemeinerte Szenarien, interpretiert.

Wir leiten konstruktive Ergebnisse zu Good-Deal-Bewertung und -Hedging mittels BSDEs her. Die Ergebnisse werden sowohl im Rahmen von Prozessen mit Sprüngen mit unvorhersehbarem Ereignisrisiko, als auch im Brown'schen Rahmen mit Modellunsicherheit erzielt. Im Falle von Sprüngen nutzen wir die Theorie zu BSDEs mit Sprüngen und liefern Beispiele in (Semi-)Markov-Modellen, die insbesondere für versicherungsmathematische Anwendungen von Bedeutung sind. Im Brown'schen Fall liefern wir neue Beispiele für konkrete No-Good-Deal-Bedingungen mit expliziten Formeln für Bewertung und Hedging, aufbauend auf klassischen Optionsbewertungsformeln (Black-Scholes, Margrabe oder Heston). Unter Knight'scher Unsicherheit bezüglich des nicht genau bekannten objektiven realen Maßes untersuchen wir hier Robustheit des Hedgings unter Berücksichtigung der Abneigung des Investors gegen Ungewissheiten. Bei Annahme mehrerer Referenzmaße als Kandidaten für das (unsichere) reale Maß führt ein Worst-Case-Ansatz zu Good-Deal-Hedging, welches robust bezüglich Unsicherheit, im Sinne von gleichmäßig über alle Referenzmaße mindestens im-Mittel-selbstfinanzierend, ist. Die Ergebnisse zu Drift- und Volatilitätsunsicherheiten werden separat präsentiert, wobei für erstere klassische BSDEs und für letztere BSDEs zweiter Ordnung zur Anwendung kommen. Bei Driftunsicherheit zeigen wir außerdem Existenz eines Worst-Case-Maßes unter dem sich Bewertungen und Hedging wie bei Abwesenheit der Unsicherheit berechnen lassen. Hier liefert der Robustheitsansatz, dass bei hinreichend großer Driftunsicherheit Good-Deal-Hedging äquivalent ist zur Risikominimierung. Im Falle von Volatilitätsunsicherheit legen wir ein Beispiel für Put-Optionen in einem Black-Scholes-artigen Modell mit unsicherer Volatilität vor, in dem explizite Lösungen zur (robusten) Good-Deal-Bewertung und Hedging unter einem Worst-Case-A-priori-Maß berechnet werden können.

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À mes deux amours, Darlène et Keynane.

Contents

Introduction

This thesis is concerned with *backward stochastic differential equations (BSDEs)* and with hedging and valuation of contingent claims in incomplete financial markets. BSDEs have by now found numerous applications in mathematical finance, where they have proved to be suitable tools in describing solutions to many stochastic optimization problems of practical relevance. BSDEs form a common theme for all chapters of the thesis, and will be used throughout in different forms depending on the problem at hand. In particular, Chapter 1 of the thesis is concerned with theoretical foundations of *BSDEs with jumps* (in short *JBSDEs*), which are BSDEs driven jointly by a Brownian motion and a random measure. We study wellposedness (existence and uniqueness) and comparison for bounded solutions to this class of BSDEs, for jumps that may have infinite activity with compensators being possibly time-inhomogeneous. Moreover in this chapter, applications of the JBSDE theory will be presented dealing with the utility maximization problem in finance. The remaining chapters of the thesis (Chapters 2 to 4) deal overall with the problem of valuation and hedging of contingent claims in incomplete financial markets. Valuation and hedging are classical topics in mathematical finance for which many approaches have been studied in the literature, especially in the context of incomplete markets where some risks may not be perfectly hedgeable, and valuation and (partial) hedging may involve solving stochastic optimization problems. As far as this thesis is concerned, we will focus on the no-good-deal approach, which does not only prevent arbitrage opportunities from the market, but also excludes an economically meaningful notion of deals that are "too good". This leads to so-called *good-deal valuation bounds*, to which a corresponding concept of hedging will be associated. In a general framework with no-good-deal constraints defined in terms of abstract *correspondences* (set-valued mappings) for the pricing measures, we will obtain results on good-deal hedging and valuation in terms of solutions to BSDEs. We will provide examples with explicit formulas that facilitate computations, for specific correspondences associated to more concrete no-good-deal constraints. In particular in Chapter 2 we will apply our theoretical results on JBSDEs from Chapter 1 to good-deal valuation and hedging in a setup allowing for unpredictable event-risk, which is modelled by a discontinuous filtration supporting a random measure and a Brownian motion simultaneously. Another topic of central interest in this thesis is robustness. In general, a robust concept will refer to one which remains effective under different admissible market scenarios/variables. In the presence of *model uncertainty* (ambiguity), the scenarios will correspond to the uncertain priors (models) and we will analyze in Chapters 3 and 4 robust concepts of good-deal valuation and hedging with respect to model uncertainty. We will focus on continuous filtrations, which will allow us to use the classical theory of BSDEs driven solely by a Brownian motion for the case of uncertainty about the excess return of traded assets (cf. Chapter 3), and the theory of *second-order BSDEs* (shortly *2BSDEs*) for the case of uncertainty about the volatility (cf. Chapter 4). Before giving a more detailed account of the contributions of the thesis, we next explain the necessary background on BSDEs, valuation and hedging in incomplete markets, and model uncertainty. These three general themes are central to the thesis and their connections with different chapters will also be made more precise.

An overview of the theory of backward SDEs

BSDEs are studied and used intensively in this thesis. To relate the contributions of the thesis to the historical developments of BSDEs, we present a short overview of advances in the theory that culminated in a wide range of applications to optimal control problems in mathematical finance. Classical BSDEs form a class of stochastic differential equations (SDEs) of the type

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad t \in [0,T],$$

They are described by a semimartingale dynamics for which a terminal condition ξ is given (instead of an initial one as for forward SDEs), and the generator function f (drift of the dynamics) varies with the value process Y of the equation and the control process Z integrated by the driving Brownian motion W under its natural filtration $(\mathcal{F}_t)_{t\leq T}$. The solution of a BSDE consists of the couple (Y,Z). Originally BSDEs appeared in [Bis73] with linear generators functions. [PP90] were the first to study existence and uniqueness of square integrable solutions in the classical setting for BSDEs under global Lipschitz assumptions on the generator. Such BSDEs will appear in Chapter 3, under a uniformly boundedness assumptions on the no-good-deal constraint correspondence for the Girsanov kernels of pricing measures. For a detailed exposition on applications of classical BSDEs in mathematical finance and additional results including a comparison principle, we refer to [EPQ97]. Beyond the Lipschitz setting, notable extensions include the case of generators with quadratic growth in the Brownian integrand Z for which [Kob00] has studied bounded solutions (see also [Tev08, BE13]). This has found crucial applications in utility maximization in incomplete markets initiated by [RE00] and [HIM05]. It has been shown in [DHB11] that BSDEs with generators that are of super-quadratic growth are typically illposed. Beyond quadratic growth and with generators that are only convex, [DHK13, DHK15] proved existence and uniqueness of minimal supersolutions relying on compactness rather than fixed-point arguments. This solution concept will be used in Chapter 3, where we will consider no-good-deal constraint correspondences that are not necessarily uniform bounded. Let us mention that by now there exists a plurality of numerical methods for simulation of BSDEs, including Monte-Carlo methods which are particularly relevant for higher dimensional problems. For advances in this direction, we refer to [BT04, GLW05, BD07, GT15, BT14].

BSDEs that are driven not only by a Brownian motion W but additionally by a random measure are shortly referred to as JBSDEs (i.e. BSDEs with jumps) and involve a second stochastic integral with respect to the compensated random measure. Their dynamics are of the form

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s)\, ds - \int_t^T Z_s\, dW_s - \int_t^T\!\!\int_E U_s(e)\, \tilde{\mu}(ds, de), \quad t \in [0, T],$$

with $\tilde{\mu} = \mu - \nu^P$ denoting the compensated random measure of some integer-valued random measure μ on a space E for a stochastic basis $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\leq T}, P)$. The solution of a JBSDE is now a triple (Y, Z, U), where the jump integrand U lives in a possibly infinite-dimensional function space and also appears in the generator of the BSDE. For such JBSDEs, [TL94, BBP97] studied square integrable solutions under global Lipschitz conditions in a time-homogeneous setting for Poisson random measures. Bounded solutions to JBSDEs have been studied in [Bec06] for a random measure that is possibly inhomogeneous in time but of finite jump activity, covering a family of generators that satisfy a certain monotonicity property but need not be (globally) Lipschitz in the jump integrand, see also [Par97, Roy06]. A similar study will be considered in Chapter 1 but for possibly infinite activity of jumps, and increased degree of complexity of the generators also allowing for a comparison principle for such JBSDEs. Indeed it appears here (see also [BBP97, Roy06, CE10]) that comparison principles for JBSDEs require more delicate technical conditions than in the Brownian case. These comparison principles will be applied in Chapter 2 to derive JBSDEs for good-deal valuation bounds and associated hedging strategies. We will consider in Chapter 1 applications of our JBSDE theory to the utility maximization problem in finance, for jumps of infinite activity. Note that JBSDEs with generator of quadratic growth in the Brownian integrand have been studied for a particular generator and infinite activity of jumps in [Mor09, Mor10], in [KTPZ15a] also under time-inhomogeneity, and in [EMN14] in general under finite activity assumptions. For numerical analysis of JBSDEs, see e.g. [BE08].

There is a strong connection between BSDEs and the theory of partial differential equations (*PDEs*). In fact (first-order) Markovian BSDEs for which the generator additionally depends on the solution of a forward SDE, hence referred to as forward-backward SDEs (alternatively *FBSDEs*), are probabilistic representations à la Feymann-Kac for second-order quasi-linear PDEs (i.e. PDEs involving only a linear dependency in the Hessian of the solution). Indeed, the PDE terms depending on the second-order derivative of the solution can only arise from the quadratic variation of the forward process via Itô's formula. Probabilistic representations of PDEs pave the way to numerical Monte-Carlo schemes for simulation of their solutions, which again are more relevant for PDEs with high dimensional state-space. Note that in the case of Markovian JBSDEs, an additional integral appears in the formulation of the PDE, hence yielding a partial-integro differential equations (PIDEs). Due to their importance in practice, one would also like as for quasi-linear PDEs to have a probabilistic representation for fully nonlinear PDEs (i.e. PDEs involving a nonlinear dependency in the Hessian of the solution), which are an important class containing e.g. *Hamilton-Jacobi-Bellman (HJB)* equations. It is exactly this fact that motivated [CSTV07] to formally introduce the notion of 2BSDE originally in connexion to the solution to the second order stochastic target problem first introduced by [ST09]. To ease exposition, a 2BSDE in its simplified form is an equation of the type

$$Y_t = \xi - \int_t^T \frac{1}{2}\Gamma_s \widehat{a}_s - H_s(Y_s, Z_s, \Gamma_s)\, ds - \int_t^T Z_s dB_s + K_T - K_t, \quad P\text{-a.s., } t \in [0,T],\ \forall\, P \in \mathcal{P}_H$$

where $\widehat{a}$ is the (ω-wise) density of the quadratic variation of the coordinate process B on the canonical Wiener space of continuous paths, $\mathcal{P}_H$ is a subset of (typically mutually singular) local martingale measures for B, and K is a non-decreasing process with $K_0 = 0$. Note that contrary to classical BSDEs, the dynamics of 2BSDEs is required to hold almost-surely under P, for all P in a family $\mathcal{P}_H$ of reference probability measure, that is to say *quasi-surely* with respect to $\mathcal{P}_H$. In this form the solution to the 2BSDE is the triple (Y, Z, Γ), and the generator $\widehat{F}$ is the convex conjugate of a nonlinear function H in its third argument Γ, satisfying $\widehat{F}_t(Y_t, Z_t) := \frac{1}{2}\Gamma_s \widehat{a}_s - H_s(Y_s, Z_s, \Gamma_s)$. For classical BSDEs, the solution components Y and Z correspond to the PDE solution and its gradient (first-order derivative) respectively. For 2BSDEs in a Markovian setting with a canonical forward process, one has an additional unknown variable Γ in the dynamics of the BSDE which essentially corresponds to the Hessian (second-order derivative) of the solution to a fully nonlinear PDE (justifying the appellation “second-order” BSDE). For first applications of 2BSDEs in mathematical finance, let us mention among many others [ÇST07, ST09] for the super-replication problem under Gamma constraint and [MPZ15] for the robust utility maximization under volatility uncertainty [ALP95, Lyo95]. In Chapter 4, we extend the list of applications in finance by using 2BSDEs to describe the solutions to good-deal valuation and hedging problem that are robust (in some sense to be made precise later) with respect to volatility uncertainty. The original formulation of 2BSDEs in [CSTV07] was in a Markovian setting and is somewhat different to the one presented above. The above is a particular case of [STZ12] who used the quasi-sure analysis of [DM06] to obtain a general formulation for possibly non-Markovian 2BSDEs and obtained a wellposedness theory for 2BSDEs with Lipschitz generators. Note that in the language of G-stochastic calculus of [Pen10], wellposedness of 2BSDEs with zero generators can be viewed as a martingale representation theorem for G-martingales (and G-expectations in particular). The wellposedness theory was later extended by [PZ13] to 2BSDEs with quadratic generators. Subsequently, [MPZ13] and [KTPZ15c, KTPZ15b] studied 2BSDEs reflected on an obstacle and 2BSDEs with jumps respectively. Some numerical schemes for 2BSDEs based on Monte-Carlo or/and finite difference methods have been suggested in the literature, e.g. in [CSTV07, FTW08, GZZ15, PT14]; see also [BET09] for a survey on the probabilistic numerical methods for nonlinear PDEs in general.

Hedging and pricing approaches in mathematical finance

Investing in financial markets involves facing some risk that can be synthesized either perfectly (one speaks of replication) or only partially by dynamic trading in liquid assets. The seller of a financial contract (contingent claim) is usually confronted with the following problem: what valuation would she like to sell the claim to the buyer for, to enable a certain form of hedging against the risk of loss at delivery of the claim? A financial market where all contingent claims can be replicated is referred to as *complete*. The significance of such markets lies in that they allow for pricing by replication so that under the viability of the market the price of a contingent claim is the cost of the replicating portfolio. This was the insight behind [BS73, Mer73] where assuming that asset prices follow a geometric Brownian motion, the authors obtained the price of vanilla call/put options by replicating with the delta-hedging strategy and deriving the celebrated Black-Scholes formula for option pricing. The Black-Scholes formula has been extended for pricing other types of options; for instance, the Margrabe formula [Mar78] is used to price European exchange options, i.e. options to exchange one risky asset for another at a pre-specified maturity time. The Black-Scholes and Margrabe formulas will play a role in the examples of Chapter 3 (see also example at the end of Chapter 4), where we will derive good-deal valuations and hedges via these formulas for incomplete models with traded and non-traded assets. The prices resulting from replication are preference-free and can be computed by taking the expectation of the option's discounted payoff under an *equivalent martingale measure* (also called risk-neutral measure). The latter is a probability measure equivalent to (i.e. with exactly the same null-sets as) the real-world probability measure and under which asset prices and associated wealth processes discounted at the bank account's interest rate are martingales, i.e. at each time, the present value is the best prediction for future values given past information. In other words under an equivalent martingale measure, risky assets have zero excess returns, i.e. same mean-return as with the riskless asset, e.g. as in the Black-Scholes model. The connection between risk-neutral pricing and martingale theory was first put into rigorous mathematical perspective by [HK79, HP81]. They characterized absence of arbitrage (also called free lunch) in a market with discrete-time trading by the existence of an equivalent martingale measure, their result is now known as the fundamental theorem of asset pricing. A consequence of this theorem and the classical predictable representation property of martingales is that completeness of the financial market is equivalent to uniqueness of the equivalent martingale measure. These results were later generalized to continuous-time trading by [DS94, DS98], in the context of asset prices being semimartingales. The latter publications introduced a reasonably general notion of market viability, namely the *no-free lunch with vanishing risk* (abbreviated *NFLVR*) condition, and linked it rather to a martingale (resp. local martingale, sigma-martingale) property of bounded (resp. locally bounded, unbounded) asset prices.

In this thesis, we concentrate on pricing and valuation in the context of incomplete market where indeed many financial claims carry some inevitable risk. This may include, for instance, claims that are contingent on some non-tradeable underlying assets, e.g. weather derivatives or some volatility derivatives. Transaction costs, jumps in the underlying asset price, or unpredictable events in the information are possible reasons for incompleteness of a financial market. Convexity of the set of equivalent martingale measures implies that incomplete arbitrage-free markets admit infinitely many pricing measures; yielding notably an interval of risk-neutral prices for non-replicable claims. The issuer of a financial contract striving for robustness with respect to price misspecification and preference-freeness may therefore wish to sell at the upper bound over all possible no-arbitrage prices, so-called *upper no-arbitrage bound*. The buyer's ideal valuation can be interpreted analogously as the corresponding lower bound. This valuation approach in incomplete markets was rigorously introduced by [EQ95], who showed that the upper no-arbitrage bound is exactly the minimal capital that allows the seller to super-replicate the claim in almost every state of the world by dynamically trading in the risky assets. The cost of *super-replication* being given by the supremum over all the risk-neutral prices, the corresponding wealth process was shown to be a supermartingale under any equivalent martingale measures and the associated super-replicating strategy can be derived via the optional decomposition theorem (see [Kra96, FK97, FK98]). Super-replication indeed is an extremely safe concept of hedging, since it excludes the possibility of losses and, while eventually allowing even for intermediate consumption, still ensures the terminal wealth to dominate the liability of the investor at delivery. Unsurprisingly, super-replication is often too costly and may not be appropriate for some practical applications; the minimal capital it requires for instance may be too high to find a buyer.

For a long time, research in mathematical finance has been investigating about alternative approaches to (partial) hedging that require lower capital than super-replication would, hence making it more likely to find a buyer. Apart from the no-good-deal approach whose background we describe in the next paragraph and which lies at the heart of Chapter 2, 3 and 4, various solutions have been suggested in this regard: The *quantile hedging* approach of [FL99] allows the seller of a claim to charge a smaller amount to the buyer but still be able to dominate his liability with some target confidence level. This is more a hedging approach than a pricing one, since its main objective is to limit the risk of loss for the seller to a maximal pre-specified level by requiring a minimal capital that will make her position acceptable in this sense. In the same direction, one can consider *risk minimization* and *mean-variance hedging* whose objective are the minimization of a quadratic functional of the tracking (hedging) error of trading strategies. In order to achieve this for risk-minimization, one relaxes the self-financing requirement of the replicating strategy (corresponding to vanishing tracking error) and instead requires a notion of *mean-self-financing* strategy that corresponds to a martingale property of the tracking error. Risk minimization was first introduced by [FS86] in the situation where asset prices are modelled directly as martingales, and later extended in [FS90] to the general semimartingale case where it could only be defined in a local sense (local risk-minimization). The local risk-minimizing strategy was ultimately derived via the so-called *Föllmer-Schweizer decomposition* of the wealth process, which can be viewed as a generalization to the semimartingale case of the well-known Galtchouk-Kunita-Watanabe decomposition from martingale theory. Naturally, a pricing concept is attached to risk-minimization via the so-called *minimal martingale measure*. However hedging according to some quadratic criterion has been criticized, mainly because it penalizes gains and losses in the same way. We will provide (cf. Chapter 3) a potential argument against this criticism by showing that if drift uncertainty is sufficiently large in the market, then risk-minimization coincides with robust good-deal hedging (in a suitable sense); the latter is using a non-quadratic hedging criterion by minimizing a dynamic coherent risk measure.

If one rather insists on the self-financing property of the hedging strategy, then a quadratic hedging criterion leads to mean-variance hedging as studied in [BL89, DR91]. Also here one obtains a valuation that is consistent with the hedging criterion and can be computed under the *variance-optimal martingale measure*. A comprehensive survey of both approaches can be found in [Sch01]. However it turns out that the minimal and variance-optimal martingale measures may be only signed measures in general, and hence may lead to negative prices for some positive claims. This is clearly an undesirable feature of these two valuation approaches. However, for a more specific Markovian (incomplete) model of the stock price, for instance the Heston stochastic volatility model [Hes93], the minimal martingale measure can be written explicitly in terms of the market price of risk. In the Heston model the squared volatility process is a Cox-Ingersoll-Ross (CIR) process and the price of European call/put options under the minimal martingale measure is given by the Heston formula which is explicit up to the computation of a one dimensional improper integral. Using a single risk-neutral measure for valuation is clearly not conservative, as it introduces mark-to-market risk that can accumulate due to the necessary regular calibration. In Chapter 3, we suggest a more conservative approach in an example that shows how a robust valuation (and hedging) of volatility risk over a family of risk-neutral measures in the Heston model can be obtained, by restricting the mean-reversion level of the variance process to be within some confidence interval.

Let us mention here also that a further alternative solution to the limitations of the above-mentioned approaches is to take into account some utility-related preference of the investor or her aversion towards risk; this leads to rational pricing and hedging concepts that are consistent with the maximal expected utility of the investor. The literature in this direction is quite developed, and generally distinguishes between two approaches: the utility-indifference approach (cf. [HH09] for an overview and further references), and the utility-based approach (cf. [Dav97, HK04]). We do not say more about these approaches. Instead, we will present in Chapter 1 an example, for illustration of our JBDSE theory therein, that will deal with the solution to the classical expected utility maximization problem in incomplete market with additional liability. Note that this approach, somewhat problematically, assumes precise knowledge of the objective real-world probabilities. This is restrictive since model uncertainty, in particular about (highly uncertain) drift and volatility parameters under the real-world measure, is a problem in itself for practical applications. Good-deal valuation and hedging in the presence of model uncertainty will be studied in Chapters 3 and 4.

In this thesis, we are mainly interested with the so-called *no-good-deal approach* to valuation of contingent claims in incomplete markets; cf. Chapter 2 to 4. As mentioned before, recall that no-arbitrage bounds are typically too large for most practical applications involving non-replicable claims. The no-good-deal approach is a fairly conservative one that lies between using a single measure for pricing and using all equivalent martingale measures. Indeed the main idea is to obtain tighter valuation bounds, called *good-deal bounds*, by using as pricing measures only a subset of the equivalent martingale measures preventing some economically meaningful notion of *good deals*. The latter could be interpreted as trading opportunities that are too favorable and therefore should also be excluded from the market. Inherent to the concept of good-deal valuation is therefore already a certain notion of robustness (namely with respect to the smaller set of pricing measures as generalized scenarios).

Good-deal bounds were introduced by [CR00] mostly in discrete time, interpreting good-deals as trading opportunities that admit an instantaneous mean excess return per unit volatility risk (called *instantaneous Sharpe ratios*) above a certain threshold. Their no-good-deal constraint therefore was imposed as a bound on the instantaneous Sharpe ratios in a financial market that is extended by additional price processes for derivatives. Their results were rigorously extended to continuous time by [BS06] in a Markovian model of asset prices and additional factor processes possibly exhibiting jumps. Using the so-called Hansen-Jagannathan (HJ) bounds (see [HJ91]), both papers showed that the constraint on the instantaneous Sharpe ratios can be obtained by pricing only under equivalent martingale measures satisfying a bound on the norm of their *Girsanov kernels*. The HJ bounds basically show that the maximal Sharpe ratio over all portfolio strategies cannot exceed the ratio of the standard deviation of a stochastic discount factor (i.e. the Radon-Nikodym derivative of the pricing measures) to its mean. In continuous (Brownian) filtrations, imposing a bound on instantaneous Sharpe ratios is basically equivalent to imposing a bound on the optimal expected growth rates [Bec09]. Such local no-good-deal constraints for pricing measures are favorable for good time-consistency properties of the resulting good-deal bounds; cf. [KS07b]. Following this remark, we will first consider in Chapters 2 and 3 a general theory of good-deal valuation and hedging for local constraints on Girsanov kernels given in terms of abstract correspondences. This will provide some flexibility as far as the choice of the no-good-deal constraint is concerned (e.g. in the jump setting of Chapter 2 where the Sharpe ratio constraint is no longer equivalent to the optimal growth rate one), but also will prove necessary in the presence of uncertainty, cf. Chapter 3, where the aggregate no-good-deal constraint under uncertainty may be different from any classical one. Note that good-deal bounds have also been defined by some notion of expected utilities [CH02, Cer03, KS07b].

Good-deal theory has been developed for a long time as a pure valuation approach (see [BY08, BL09, MMM13, Mur13] in a Brownian setting and [BS06, KS07b, Don11] in a setting with jumps). Contributions about hedging only appeared recently, mostly in the setting of a Brownian filtration. These started from [Bec09] who uses classical BSDEs to derive hedging strategies as minimizers of dynamic coherent risk measures [ADE$^+$07] of no-good-deal type yielding the good-deal bound as the minimal capital for acceptability, i.e. the market consistent risk measure in the spirit of [BE09]. [CT14] studied mean-variance hedging in the context of good-deal valuations and concluded that both hedging approaches perform reasonably well. Throughout this thesis, we follow the good-deal hedging approach of [Bec09], for which valuations and hedges will be described by different classes of BSDEs (classical BSDEs, JBSDEs, 2BSDEs), depending on the framework in use. We note that hedging by minimizing a certain risk measure that allows for market consistent valuation is by now standard in the literature [CGM01, BE05, KS07a, BE09]. In addition, dynamic risk measures in general are well-connected to BSDEs; cf. [Ros06, PR15].

Robustness and model uncertainty in finance

Robustness and model uncertainty are important topics in finance and decision theory; cf. [Con06, HS01]. Since definitions of the no-good-deal constraints involve the objective real-world probability measure, model uncertainty is also relevant to good-deal theory. Chapters 3 and 4 are concerned with robust approaches to uncertainty, in the *Knightian* sense (cf. [Kni21]), about the objective probability measure with respect to which good deals are defined, and good-deal bounds and hedging strategies are computed. In economic theory, it has been argued that incorporating uncertainty aversion provides a theoretical ground for explaining some behavioral observations such as the famous Ellsberg paradox [Ell61] or the equity premium puzzle [MP85]. Uncertainty in financial markets is a serious concern for (typically ambiguity-averse) investors who permanently strive for robustness in the valuation and hedging of their financial risks. Diverse mechanisms have been elaborated in the mathematical finance literature to take into account aversion towards uncertainty in financial modeling. In this thesis, we use a *multiple prior* approach to robustness under uncertainty proposed by [GS89, CE02], where an uncertainty-averse investor or decision-maker seeks to protect herself against an eventual misspecification of probabilities by considering the most conservative (worst-case) line of action with respect to some confidence region of subjective probability measures called priors. The mathematical finance literature in this direction is wide and essentially distinguishes between *drift uncertainty* and *volatility uncertainty*. Following the same distinction, we will first consider drift uncertainty in Chapter 3 and then volatility uncertainty in Chapter 4, both in the context of good-deal theory.

Drift uncertainty englobes in particular uncertainty about the market price of risk of traded assets, which naturally embeds into a setup where priors are equivalent to each other, i.e. they share the same nullsets and therefore agree about the impossible events in the market. This framework has been considered for instance in [DW92, Que04, GUW07, Sch08] for solving the maxmin expected utility maximization problem. In Chapter 3 we study robustness of good-deal hedging strategies for a worst-case approach to good-deal valuation, which yields larger good-deal bounds under uncertainty. We show the existence of a worst-case prior under which dynamic valuations and hedges can be computed as in the absence of uncertainty. For our results under drift uncertainty, we rely on classical BSDE methods under a fixed reference prior to which all others are equivalent. In the case of volatility uncertainty in Chapter 4, priors may no longer be equivalent to each other and we use 2BSDEs instead, for deriving valuation and hedging results.

Historically, [ALP95, Lyo95] introduced the *uncertain volatility model* as a model of stock prices in the presence of volatility uncertainty, in which pricing and hedging of contingent claims in incomplete markets can be done in an analog way as in the (complete) Black-Scholes model. Typically, priors in the uncertain volatility model are mutually singular, since they may have disjoint supports; see e.g. [DM06, EJ13, EJ14]. This model is by now standard in the literature, and consists in modeling asset price dynamics under risk-neutral measures on the path-space that, being viewed as subjective priors, are parametrized by different volatility processes taking values in a pre-specified confidence interval of volatility values. [ALP95, Lyo95] (see also [Vor14] for a model of stock prices as geometric G-Brownian motions) derived no-arbitrage valuation bounds for financial derivatives in terms of the solution to a fully nonlinear PDE called the Black-Scholes-Barenblatt equation, which is a nonlinear analog of the Black-Scholes PDE in the presence of volatility uncertainty. In particular for convex payoff functions they showed that the worst-case model for the upper valuation bound corresponds to the highest volatility under which the two pricing PDEs coincide. In general when priors are non-dominated (i.e. they may disagree about the impossible market scenarios), one has to resort to different techniques for dynamic formulations and solutions of robust stochastic optimization problems; see e.g. [EJ14]. A typical difficulty in this case appears when defining the essential supremum of a family of random variables, which in the dominated case is well-defined up to a null set for a dominating prior. However if the priors are mutually singular, the definition of essential supremums necessitates some aggregation procedures for the null-sets of priors, which can then be disjoint. The quasi-sure analysis of [DM06]) provides a suitable framework for dealing with these technical issues, and is used for example in [DK13a, MPZ15] for maxmin expected utility maximization under volatility uncertainty. For our 2BSDE approach in Chapter 4, the quasi-sure analysis will be used naturally following the wellposedness theory of Lipschitz 2BSDEs in [STZ12]. In this chapter, robustness of good-deal hedging strategies for worst-case good-deal valuation under volatility uncertainty will be shown. Due to the technical issues mentioned above, we are not able to show existence of a worst-case prior for dynamic good-deal bounds in the general theory. However, in an example for European put options in a (two dimensional) Black-Scholes model for a traded and a non-traded asset, we will constructively identify a worst-case prior for dynamic valuations which mimics the relation to convexity of the payoff function as in [ALP95, Lyo95, Vor14]. A closed-form expression of the robust hedging strategy will be subsequently given, after explicitly identifying the solution to the 2BSDE in the example.

Let us mention that recent developments in robust finance include the *drift-and-volatility uncertainty* framework of [EJ13, EJ14] for formulation of the pricing, hedging and maxmin expected utility maximization problems in a continuous dynamic setting, taking into account the investor's uncertainty about both volatility and drift. Solutions to the robust utility maximization problem under drift-and-volatility uncertainty in continuous time have been investigated recently by [BP15] using PDE methods, focusing on ellipsoidal drift-uncertainty for each fixed volatility scenario. Although dealing only with drift uncertainty, Chapter 3 will consider some cases where the confidence set of drift uncertainty is also described by an ellipsoid, which seems natural for drift uncertainty modeling; cf. also [GUW07]. Note that [Nut14] also recently showed existence of an optimal trading strategy for the maxmin utility-maximization problem, for arbitrary sets of priors and bounded utility functions, but restricting his analysis to discrete time. Drift-and-volatility uncertainty is however not the route followed in this thesis, as both types of uncertainty will be considered separately. This is partly motivated by the fact that in our dynamic setting standard conditions for wellposeness of 2BSDEs (in particular regularity and convexity of the generator) as in [STZ12] may not hold for the dynamic good-deal valuation and hedging problem in Chapter 4 if one considers drift uncertainty in addition.

Contribution of the thesis

This thesis is organized in four chapters which are mostly self-contained and can be read almost independently. The connections between the chapters' results can be specified as follows: Chapter 1 deals with a theoretical study of wellposedness and comparison for solutions to BSDEs with jumps of infinite activity and time-inhomogeneous compensators; Chapters 2 to 4 are concerned with good-deal valuation and hedging. More precisely, Chapter 2 applies some results of Chapter 1 to market models that allow for jumps described by abstract random measures; Chapters 3 and 4 finally study robustness (of valuation and hedges) with respect to uncertainty about the drift of traded assets for the former and about their volatility for the latter. A more detailed chapter-wise description of the contribution of the thesis will now be given below.

Chapter 1: Concrete criteria for wellposedness and comparison of BSDEs with jumps of infinite activity

This chapter is based on [BBK15] and studies bounded solutions (Y, Z, U) to JBSDEs. Recall that in comparison to classical BSDEs which are driven only by a Brownian motion W, such JBSDEs are additionally driven by a random measure $\tilde{\mu}$ and involve a second stochastic integral with respect to the compensated random measure $\tilde{\mu} = \mu - \nu^P$ with integrand U on which the generator $f(Y_-, Z, U)$ may also depend.

We extend the analysis of JBSDEs beyond classical Lipschitz assumptions (cf. [TL94, BBP97]) on the generator f by concentrating on a family of generators that satisfy a certain monotonicity property, but do not need to be globally Lipschitz in the U-component; see also [Bec06, Roy06]. We do not require the compensator $\nu(dt, de)$ of $\mu(dt, de)$ to be a product measure like $\lambda(de) \otimes dt$, as it would be natural for random measures of jumps of Lévy type for instance. Instead, ν is only assumed to be absolutely continuous to some reference product measure $\lambda \otimes dt$ with a bounded Radon-Nikodym derivative ζ (implying time-inhomogeneity) where λ is a σ-finite measure, hence allowing for infinite activity of the driving jumps. This provides wide scope for stochastic dependencies between W and $\tilde{\mu}$, which can be relevant in applications; cf. examples in [BS05]. Furthermore, it embeds a range of interesting driving processes for BSDEs in addition to Brownian motion, including Lévy processes, Poisson random measures, marked point processes, Markov chains or much more general step processes (as in [HWY92], Chapter 11, including e.g. semi-Markov processes), connecting our analysis to research from [NS01, CE10].

As usual in the BSDE literature, we require a key property on the filtration, namely that $\tilde{\mu}$ and W together have the weak predictable representation property for martingales. In order to deal with the time-inhomogeneous setting, we slightly extend a general but technical comparison theorem on JBSDE from [Roy06] by using a more general (A_γ)-condition, in order to derive sufficient conditions for comparison which are easier to verify, since they are formulated in terms of concrete properties of generator functions from our family of interest. This gives rise to a-priori estimates of the L^∞-norm of the Y-component of the JBSDE solution. Additionally we obtain existence and uniqueness results for bounded JBSDE solutions (i.e. wellposedness of such JBSDEs) in the case of jumps with finite activity, as in [Bec06]. These steps enable us to advance to the case of infinite activity. To this end, we first approximate the generator by truncating the activity of the jumps using the σ-finiteness of λ. This leads to a monotone sequence of generators for which solutions do uniquely exist. Then using monotone stability arguments like in [Kob00] enables us to obtain wellposedness for the initial JBSDE generator. However, it turns out that such arguments only work at first for terminal conditions ξ which are small in L^∞-norm. By pasting solutions for sufficiently small terminal conditions one can show convergence to the bounded solution of the JBSDE for the original data (ξ, f). For this purpose, we follow the iterative idea of [Mor10] who focused on a particular generator, and elaborate the proof slightly differently and more compactly for our general setting. For results in related but different directions, we mention [CM08] for comparison of JBSDEs with (doubly) reflection for Lipschitz generators, [KMPZ10] for (minimal) solutions with constraints on jumps for Poisson random measures of finite activity, [DI10] for time delayed generators and [KTPZ15a] for a pasting argument for quadratic JBSDEs following the fixed-point approach of [Tev08].

We note that our results are mainly stated for generators that are Lipschitz continuous in the Z-component. For results on generators of quadratic growth in Z we refer e.g. to [Mor10, EMN14, KTPZ15a]. [EMN14] work in finite activity and [Mor10] considers a particular generator function arising from a specific utility maximization problem. As in our setup [KTPZ15a] also works in infinite activity and considers time-inhomogeneous compensators. However, the applicability of their results may be less straightforward than ours due the abstract nature of their assumptions on the generator which are stated in terms of existence of some abstract processes satisfying strong integrability requirements such that certain non-linear estimates hold without exception of a null set. Another contribution of this chapter is therefore that we provide concrete conditions on generators that are easier to verify in applications. For illustration purposes, we apply our results to the utility maximization problem in finance, for power and exponential utility functions complementing results e.g. from [HIM05, Sek06, Bec06, Mor10, Nut12a]. In Chapter 2, a nonlinear example on good-deal valuation and hedging in incomplete markets with jumps will be also covered.

Chapter 2: Hedging under generalized good-deal bounds in jump models with random measures

In this chapter, which is based on [BK15c], we study good-deal hedging and valuation in general jump models driven by random measures. More precisely, we suppose the presence of unpredictable event-risk in the market in the sense that the information flow (filtration) may be discontinuous, allowing for non-trivial purely-discontinuous price processes with totally inaccessible jump times. [BS06] first considered good-deal valuation in a Markovian setting with jumps, where good-deal bounds were defined by a constraint on the instantaneous Sharpe ratios and derived subsequently as solution to HJB equations. Although [BS06] focused only on valuation, they raised the crucial need for a hedging theory that corresponds to good-deal valuation. A first attempt to good-deal hedging was by [Bec09] in a Brownian setting, who derived hedging strategies from minimizing suitable dynamic coherent risk measures that allow for optimal risk sharing with the market through good-deal valuation. Another attempt, still in a Brownian setting, is based on a quadratic hedging criterion like mean-variance hedging and was developed by [CT14]. In the jump setting, [Del12] studied the approaches of [Bec09, CT14] for point-processes with state-independent jump intensities, restricting to a market with two risky assets (a traded and a non-traded one) and considering solely Sharpe ratio no-good-deal constraints. For a related problem, we study in this chapter a generalization to multiple non-necessarily Markovian risky assets, more general jump processes driven by abstract integer-valued random measures and generalized no-good-deal constraints on Girsanov kernels of pricing measures parametrized by correspondences (i.e. set-valued mappings as studied in [AF90]).

Using correspondences provides an abstract framework for incorporating no-good-deal restrictions of different natures (e.g. instantaneous Sharpe ratios, optimal growth rates, instantaneous Sharpe ratios under uncertainty about jump intensities, etc.). We derive generalized good-deal valuation bounds for possibly path-dependent contingent claims using classical JBSDEs instead of HJB equations (under Markovian assumptions) as in [BS06, Don11, Mur13]. We first consider the case of uniformly bounded correspondences, for which the generators of the resulting JBSDEs are Lipschitz continuous. The resulting valuation JBSDE is derived from the comparison principle of Chapter 1, and its generator is the maximum of a family of linear JBSDE generators parametrized by the Girsanov kernels of no-good-deal pricing measures. This generator in general does not have an explicit form, even for the classical radial (Sharpe ratio) no-good-deal constraint. To obtain more constructive (or even explicit) expressions of the generators, we assume more structure on the contingent claim, the nature of the no-good-deal constraint or the random measure of the jumps. Examples are presented with closed-form expressions for the associated good-deal hedging strategy. The case beyond uniform boundedness of the correspondence is also considered, where the Lipschitz property of the generators is no longer ensured and approximation arguments are used.

Using a notion of good-deal hedging introduced by [Bec09], we contribute results on the existence of hedging strategies for arbitrary bounded correspondences. Moreover we obtain a characterization of the hedging strategies in terms of the solution to the JBSDE describing the good-deal valuation bound. We show that the tracking errors of hedging strategies, i.e. the dynamic difference between the good-deal bound and the profit/loss from trading, satisfy a supermartingale property under some a-priori valuation measures including the no-good-deal measures. As the martingale property of the tracking errors corresponds to hedging strategies being mean-self-financing in the terminology of [Sch01], this means that the good-deal hedging strategies can be viewed as being *at least mean-self-financing* under every a-priori valuation measure. The latter can be interpreted as a robustness property of the good-deal hedging strategy with respect to the set of a-priori valuation measures as generalized scenarios in the sense of [ADE$^+$07]. For concrete no-good-deal constraints we provide some examples where a good-deal hedging strategy can be obtained explicitly in terms of solutions to JBSDEs. Hedging is investigated only for bounded correspondences, and in the case beyond uniform boundedness we only present results about good-deal valuations.

In a discontinuous filtration, imposing a bound on the instantaneous Sharpe ratios via a bound on the norms of the Girsanov kernels of pricing measures (as in [BS06]) is not equivalent to imposing a bound on the optimal conditional expected growth rates (as in [Bec09]). We note that the growth rate constraint is mathematically less tractable than the Sharpe ratio one, at least in terms of using Lipschitz BSDEs. Indeed, it turns out that Sharpe ratio constraints fit well with the theory of Lipschitz JBSDEs for arbitrary random measures. For such constraints and for some concrete random measures of jumps, we can even obtain more simplified JBSDE descriptions of good-deal bounds and hedging strategies. In particular for jumps of a continuous-time Markov chain and without a Gaussian component, we infer from [CS12] that the JBSDE for good-deal bounds defined from Sharpe ratio constraints, for Markovian European contingent claims depending only on the terminal value of the chain, reduces to a fully-coupled system of ordinary differential equations (ODEs). The latter can be transformed (by reversing time) into an initial value problem, which can then be solved using any standard numerical ODE solver. For Sharpe ratio constraints, we also present an example for robust hedging under Knightian uncertainty about the intensity of the underlying jump process, linking the result here with those of Chapters 3 and 4. Here robustness of a hedging strategy with respect to uncertainty refers to a property of being at least mean-self-financing under every a-priori valuation measure, uniformly with respect to a family of subjective probability measures as candidates for the real-world measure and capturing the uncertainty. On the other hand, optimal growth rate constraints do not fit well with Lipschitz BSDEs since the resulting correspondence may not be uniformly bounded. For such constraints it turns out that we can still rely on the theory of Lipschitz JBSDEs when random measures with finite support of the compensator. Results are then obtained for finite-state semi-Markov processes, which are a flexible class with many practical applications, see e.g. for actuarial applications [BMS14] and references therein.

Chapter 3: Hedging under generalized good-deal bounds and drift uncertainty

This chapter is based on [BK15b] and is concerned with approaches to good-deal hedging (as in [Bec09]) under ambiguity about the objective probability with respect to which good deals are defined and good-deal bounds computed. Good-deal valuations fit into the theory of dynamic monetary convex risk measures (or monetary convex utility functionals) for which results, in particular about dual representations and time consistency, exist in high generality; see e.g. [KS07a, BNN13, DK13b]. We contribute constructive and qualitative results on the (robust, good-deal) hedging strategies, that facilitate interpretation and are accessible to computation. We pose the good-deal valuation and hedging problem in a framework with multiple priors, and follow a robust worst-case approach as in [GS89, CE02]. We note that results on good-deal valuations and hedges under uncertainty in a recent work by [BCCH14] are very different to ours. Indeed, they work mostly in discrete time and study numerical results for a different uncertainty-penalized preference functional, whereas we use dynamic coherent risk measures in continuous time and focus on rather analytical results.

After formulating a framework with predictable correspondences as in Chapter 2 but in the Wiener setting, we also describe good-deal hedging strategies and valuation bounds in terms of solutions to classical BSDEs. In the absence of uncertainty, we obtain results first for uniformly bounded correspondences and then for the case beyond uniform boundedness by approximation. Additionally we characterize the good-deal valuation bounds in the possibly unbounded case in terms of minimal supersolutions to convex BSDEs (as in [DHK13]). Notably, the abstract generalized constraints are needed to cover relevant examples of uncertainty about the market prices of risk of the assets that are available in the (incomplete) market for dynamic partial hedging. For illustration purposes, we will consider e.g. ellipsoidal correspondences which permit explicit analytic generators in the BSDEs of interest, being efficient for Monte-Carlo approximation. In general good-deal hedging strategies can comprise a *speculative* bet in the direction of the market price of risk to compensate for unhedgeable risks.

In this chapter we also provide new examples on good-deal valuation and hedging, with closed-form formulas for good-deal bounds and hedging strategies: For an exchange option between tradeable and non-tradeable assets, we give a Margrabe-type formula [Mar78] for the good-deal bound, with adjusted input parameters. For the stochastic volatility model by Heston [Hes93], we obtain semi-explicit formulas under good-deal constraints for pricing measures, which restrict the mean-reversion level of the stochastic variance process to be within some confidence interval. A graphical analysis of the dependency on model parameters is also done. An interesting aspect of the latter example shows, how a robust valuation of volatility risk (over a family of no-good-deal pricing measures) can be obtained for an absolutely continuous family of measures. To illustrate to which extend our BSDE solutions could be computed by efficient but generic Monte-Carlo methods, complementing numerical approaches to hedging from [CT14], we investigate the errors between the Monte-Carlo approximations and our analytic formula in a four dimensional example for an exchange option.

In the presence of uncertainty, we derive general results for good-deal bounds and hedging strategies that are robust with respect to uncertainty, described also by correspondences. Building on a suitable definition of good-deal bounds in the presence of uncertainty, we note that the problem with multiple priors can be related to a respective problem without uncertainty but with an enlarged good-deal constraint correspondence, which even in the most natural cases of no-good-deal restriction and uncertainty may easily not have a radial shape; hence the need of a general theory for abstract correspondences in the first place. The worst-case approach naturally leads to a robust notion of valuation by the widest good-deal bounds that are obtained over all probabilistic models under consideration. We show that there is also a notion for robust hedging, which corresponds to the aforementioned robust good-deal valuation. Indeed, there exists a unique strategy that is *robustly at least mean-self-financing*, in the sense that it is at least mean-self-financing (see Chapter 2) uniformly with respect to all priors. By saddle point arguments we derive a minmax identity, that shows how the robust good-deal hedging strategy is given by the (ordinary) good-deal hedging strategy with respect to a worst-case measure. Since we rely on BSDEs both are actually identified in a constructive manner. As intuition suggests, a robust approach to uncertainty reduces the speculative component of the good-deal hedging strategy. As a further contribution, we prove that if the uncertainty is large enough in relation to the no-good-deal constraints, then the robust good-deal hedging strategy does no longer include any speculative component, but coincides with the (globally) risk-minimizing strategy of [FS86]. This offers theoretical support to the commonly held perception that hedging should abstain from speculative objectives (see e.g. [LP00]), and moreover a new justification for risk-minimization. Finally, an example with closed-form solutions for robust good-deal bounds and hedging strategies for an option on a non-traded asset illustrates results and graphically analyzes dependencies on parameters.

This chapter has built over the Masters' thesis [Ken11] for preliminary results including those about generalized good-deal bounds in the absence of uncertainty for uniformly bounded and ellipsoidal correspondences, and part of worst-case valuation in the presence of uncertainty. All remaining results are new; among others, the examples with closed-form expressions for the good-deal bound and hedging strategy, the saddle-point results on worst-case valuation and hedging in the presence of uncertainty, and the link to risk-minimization obtained in the last section.

Chapter 4: Hedging under good-deal bounds and volatility uncertainty: a 2BSDE approach

Chapter 4 is based on [BK15a] and deals with robust good-deal hedging and valuation with respect to volatility uncertainty. We consider also here an approach under which good-deal bounds are computed as worst-case valuations over a calibrated class of priors. Contrary to drift uncertainty for which BSDE descriptions can be given under a single dominating prior (see Chapter 3), volatility uncertainty corresponds to priors that are mutually singular, and therefore necessitates a different mathematical framework for valuation and hedging results. In particular a rigorous definition of the dynamic good-deal bound as essential supremum/infimum of random variables involves additional technical care since it may not be possible to aggregate the null-sets of the different priors.

First we present some purely theoretical results on the comparison principle for solutions to 2BSDEs with different generators and terminal conditions, thus extending a result in [STZ12] for which the generators are identical. We use the so-called strong formulation of volatility uncertainty, which considers the uncertain priors as local martingale laws of stock price processes defined on the canonical Wiener space. Our definitions of worst-case good-deal bounds and hedging strategies are adapted to this framework and we follow a setup by [STZ13], which starting from the canonical space and working with regular conditional probability distributions (in short r.c.p.d.) ensures a time-consistency property (in the spirit of [NS12]) of the good-deal bounds as dynamic risk measures under volatility uncertainty. This paves the way for defining good-deal hedging again as minimization of residual risk from dynamic trading. In this chapter, we concentrate on a no-good-deal restriction imposed as a bound on the instantaneous Sharpe ratios under each reference prior separately. This provides a family of good-deal bound processes parametrized by priors, and the worst-case upper good-deal bound arises as the largest among them, i.e. their essential supremum. Building on the intuition from Chapter 3, we derive robust good-deal bounds and hedging strategies under volatility uncertainty in terms of solutions to Lipschitz 2BSDEs relying on the theory in [STZ12]. Again as in Chapter 3 robustness of the good-deal hedging strategy with respect to uncertainty is related to the property of being at least mean-self-financing uniformly over all priors. Finally, we contribute an example for put options on non-traded assets under volatility uncertainty, in which a worst-case model can be explicitly computed for the dynamic good-deal bound and closed-form formulas for robust valuations and hedges are derived, like in [ALP95, Lyo95, Vor14]. The latter works focus on robust superhedging in the presence of volatility uncertainty, whereas we focus on good-deal hedging. As an example demonstrates, our robust good-deal hedging strategy (and respective valuation bounds) can in general be very different from the super-replicating one.

1. Concrete criteria for wellposedness and comparison of BSDEs with jumps of infinite activity

In this chapter, we study JBSDEs for a specific class of generator functions that do not necessarily satisfy global Lipschitz conditions in the jump integrand. The JBSDEs in consideration are driven, additionally to a Brownian motion, by general random measures with compensators that can be inhomogeneous in time and may allow for infinite activity of the jumps of the value process. In this context, we provide in Sections 1.2 and 1.3 concrete conditions that are directly verifiable for existence, uniqueness and comparison of bounded solutions to such JBSDEs, first in the case of finite activity of the jumps and then to the infinite activity case by suitable approximations. Section 1.4 illustrates the range of applicability of our results by solving the utility maximization problem in finance for exponential and power utility functions with additive and multiplicative liability respectively. To make this chapter as self-contained as possible, we first introduce some useful notations and the mathematical preliminaries.

1.1 Mathematical framework and preliminaries

This section presents the technical framework and sets the notations. We will also summarize the key assumptions on the BSDE generator (1.7) which will play, in varying combinations, a role in our later results. First we recall essential facts on stochastic integration with respect to random measures and on bounded solutions for Backward SDEs which are driven jointly by Brownian motions and a compensated random measure. For notions from stochastic analysis not explained here we refer to [JS03] and [HWY92].

Inequalities between measurable functions are understood almost everywhere with respect to an appropriate reference measure, typically P or $P \otimes dt$. Let $T < \infty$ be a finite time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ a filtered probability space with a filtration satisfying the usual conditions of right continuity and completeness, assuming $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ being trivial (under P). Due to the usual conditions we can and do take all semimartingales to have right continuous paths with left limits, so-called càdlàg paths. Expectations under a probability Q and Conditional expectations given $\mathcal{F}_t$ are denoted by $E^Q[\cdot]$ and $E^Q_t[\cdot]$ respectively, or simply $E[\cdot]$ and $E_t[\cdot]$ when $Q = P$. Reference to the probability is omitted if clear from context. Let H be a separable Hilbert space and we denote by $\mathcal{B}(E)$ the Borel σ-field of $E := H \backslash \{0\}$, e.g. $H = \mathbb{R}^l$, $l \in \mathbb{N}$ or $H = \ell^2 \subset \mathbb{R}^{\mathbb{N}}$. Then $(E, \mathcal{B}(E))$ is a standard Borel space. In addition, let W be a d-dimensional Brownian motion. Stochastic integrals of a vector valued predictable process Z

with respect to a semimartingale X, e.g. $X = W$, of the same dimensionality are scalar valued semimartingales starting at zero and denoted by $\int_{(0,t]} Z dX = \int_0^t Z dX = Z \cdot X_t$ for $t \in [0,T]$. The *predictable* σ-field on $\Omega \times [0,T]$ generated by all left continuous adapted processes is denoted by $\mathcal{P}$ and $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(E)$ is the respective σ-field on $\widetilde{\Omega} := \Omega \times [0,T] \times E$.

Let μ be an integer-valued random measure with compensator $\nu = \nu^P$ (under P) which is assumed to be absolutely continuous to $\lambda \otimes dt$ for a σ-finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$ with some $\widetilde{\mathcal{P}}$-measurable, bounded and non-negative density ζ, such that

$$\nu(dt, de) = \zeta_t(e)\, \lambda(de)\, dt = \zeta_t\, d\lambda\, dt, \tag{1.1}$$

with $0 \leq \zeta_t(e) \leq c_\nu$ $P \otimes \lambda \otimes dt$-a.e. for some constant $c_\nu > 0$. $L^2(\lambda)$ (resp. $L^2(\zeta_t d\lambda)$) defines the space that of $\mathcal{E}$-measurable functions $\gamma : E \to \mathbb{R}$ with $\int_E |\gamma(e)|^2 \lambda(de) < \infty$ (resp. $\int_E |\gamma(e)|^2 \zeta_t(e) \lambda(de) < \infty$). Note that the Hilbert spaces $L^2(\lambda)$ and $L^2(\zeta_t d\lambda)$, are separable since the underlying measures are σ-finite and the σ-algebra $\mathcal{E}$ is countably generated (see [Coh13, Proposition 3.4.5]). Hence they admit countable orthonormal bases and are in particular Polish spaces. Since the density ζ can depend on t and ω, the compensating measure ν may be time-inhomogeneous and stochastic. This permits for a richer dependence structure between W and μ; for instance the density ζ and thereby the intensity of the jump measure might fluctuate in dependence of some diffusion process driven by W.

Let Q be a probability measures. We denote by $L^p(Q)$, $1 \leq p < \infty$, the space of $\mathcal{F}_T$-measurable random variable X with $\|X\|^p_{L^p(Q)} := E^Q[|X|^p] < \infty$, and $L^\infty(Q)$ the space of $\mathcal{F}_T$-measurable random variable $\|X\|_{L^\infty} := \|X\|_\infty = \text{ess sup}^Q |X| < \infty$. For a function $U : [0,T] \times \Omega \times E \to \mathbb{R}$ we define $|U|_\infty := \text{ess sup}_{(t,e)} |U_t(e)|$. For stochastic integration with respect to $\widetilde{\mu}$ and W we define sets of $\mathbb{R}$-valued processes

$$\begin{aligned}
\mathcal{S}^p(Q) &:= \left\{ Y \text{ càdlàg } : |Y|_p := E^Q\left(\sup_{0\leq t\leq T} |Y_t|^p \right) < \infty \right\} \quad \text{for } p \in [1,\infty), \\
\mathcal{S}^\infty(Q) &:= \left\{ Y \text{ càdlàg } : |Y|_\infty := \Big\| \sup_{0\leq t\leq T} |Y_t| \Big\|_{L^\infty(Q)} < \infty \right\}, \\
\mathcal{H}^2_\nu(Q) &:= \left\{ U \ \widetilde{\mathcal{P}}\text{-measurable} : \|U\|^2_{\mathcal{H}^2_\nu(Q)} := E^Q\left(\int_0^T \!\! \int_E |U_s(e)|^2\, \nu^Q(ds, de) \right) < \infty \right\},
\end{aligned}$$

and the set of $\mathbb{R}^d$-valued processes

$$\mathcal{H}^2(Q) := \left\{ \theta \ \mathcal{P}\text{-measurable} : \|\theta\|^2_{\mathcal{H}^2(Q)} := E^Q\left(\int_0^T \|\theta_s\|^2\, ds \right) < \infty \right\},$$

where ν^Q denotes the compensator of the random measure μ under Q. For $Q = P$, we will simply write the above-defined spaces without further mention of the underlying probability measure. As in [JS03], we define for a $\widetilde{\mathcal{P}}$-measurable function U the integral of U with respect to μ, denoted by $U * \mu$, as the optional integral process given pathwise for $t \in [0,T]$ by

$$U * \mu_t(\omega) := \begin{cases} \int_0^t \int_E U(\omega,s,e)\mu(\omega,ds,de), & \text{if finite,} \\ +\infty, & \text{otherwise.} \end{cases}$$

We recall that for any predictable function U we have $E(|U| * \mu_T) = E(|U| * \nu_T)$ by the definition of a compensator. If $(|U|^2 * \mu)^{1/2}$ is locally integrable, then U is integrable with respect to $\widetilde{\mu} = \mu - \nu$, and $U * \widetilde{\mu}$ is defined as the purely discontinuous local martingale with jump process $(\int_E U_t(e)\, \mu(\{t\}, de))_t$ by [JS03, Definition II.1.27] noting that ν is absolutely continuous to $\lambda \otimes dt$. For $Z \in \mathcal{H}^2$ and $U \in \mathcal{H}^2_\nu$ we recall that the processes $\int Z\, dW$ and $U * \widetilde{\mu} = (U * \widetilde{\mu}_t)_{t \le T}$ with $U * \widetilde{\mu}_t = \int_0^t \int_E U_s(e)\, \widetilde{\mu}(ds,de)$ are square integrable martingales ([JS03], Theorem II.1.33.). For $Z, Z' \in \mathcal{H}^2$, $U, U' \in \mathcal{H}^2_\nu$ we have for the predictable quadratic covariations of stochastic integrals that $\langle U * \widetilde{\mu}, U' * \widetilde{\mu}\rangle_t = \int_0^t \int_E U_s(e) U'_s(e)\, \nu(ds,de)$ ([JS03], Theorem II.1.33.), $\langle \int Z\, dW, \int Z'\, dW\rangle_t = \int_0^t Z_s^* Z'_s\, ds$ and $\langle \int Z\, dW, U * \widetilde{\mu}\rangle_t = 0$ by [JS03], Theorem I.4.2., using that $U * \widetilde{\mu}$ is purely discontinuous.

We denote the space of square integrable martingales by $\mathcal{M}^2$ and its norm by $\|\cdot\|_{\mathcal{M}^2}$ given by $\|M\|_{\mathcal{M}^2} = E(M_T^2)^{\frac{1}{2}}$. We recall (see [HWY92], Theorem 10.9.4) that the subspace of BMO(P)-martingales $BMO(P)$ contains any square integrable martingale M with uniformly bounded jumps and bounded conditional expectations for increments of the quadratic variation process

$$\sup_{0 \le t \le T} \left\| E_t[M_T - M_t)^2] \right\|_{L^\infty(P)} = \sup_{0 \le t \le T} \| E_t[\langle M\rangle_T - \langle M\rangle_t] \|_{L^\infty(P)} \le \text{const} < \infty$$

We will assume throughout this chapter that the Brownian motion W and the compensated measure $\widetilde{\mu}$ of an integer-valued random measure μ jointly have the weak predictable representation property (weak PRP) with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$; that means that every square integrable martingale M has a (unique) representation in the sense that

$$\text{for all } M \in \mathcal{M}^2 \text{ there exists } Z, U \text{ such that } \; M = M_0 + \int Z\, dW + U * \widetilde{\mu}\,, \tag{1.2}$$

with (unique) $Z \in \mathcal{H}^2$ and $U \in \mathcal{H}^2_\nu$. Let us note that in Chapter 2 the weak representation property will be defined as a decomposition like (1.2) for any local martingale M with integrands Z and U being integrable in the sense of local martingales. Such clearly implies the formulation above (see Section 2.1). For a literature about the weak predictable representation property we refer to [JS03],III.§4c, or [HWY92], XIII.§2. We show next how (1.2) connects with the literature.

Example 1.1. *The weak predictable representation property (1.2) (PRP) holds in the cases below. Cases 1.-4. are well known from classical theory [HWY92] (for details cf. [Bec06], Example 2.1).*

1. *Let X be a Lévy process with $X_0 = 0$ and predictable characteristics (α, β, ν) (under P). Then for $\beta \neq 0$ the continuous part X^c (rescaled to a Brownian motion) and the compensated jump measure $\widetilde{\mu}^X = \mu^X - \nu$ of X have the weak PRP with respect to the usual filtration $\mathcal{F}^X$ generated by X. An example for a Lévy process of infinite activity is the Gamma process. One could add, that weak PRP even holds in the sense Theorem III.4.34 from [JS03] for the more general class of PII-processes with independent increments. This class encompasses the more familiar Lévy processes without requiring time-homogeneity or stochastic continuity.*

2. *Assume that W and $\widetilde{\mu}$ satisfy (1.2) under P. Let P' be an equivalent probability measure with density process Z. Then the Brownian motion $W' := W - \int (Z_-)^{-1} \, d\langle Z, W \rangle$ and $\widetilde{\mu}' := \mu^X - \nu^{P'}$ have the weak PRP (1.2) also with respect to P' under the same filtration.*

3. *Let W be a Brownian motion independent of a step process X (in the sense of [HWY92], Chapter 11). Then W and $\widetilde{\mu}$, the compensated measure of the jump measure μ^X of X, have the weak PRP with respect to the usual filtration generated by X and W. An example for a step process is a multivariate (non-explosive) point process, as appearing in [CFJ14].*

4. *Furthermore, a (semi-)Markov chain X, possibly time-inhomogeneous, is also a step process. Thus weak PRP (1.2) holds for a filtration generated by a Brownian motion and an independent Markov chain, relating later results to literature [CE10, CF14] on BSDEs driven by pure-jump Markov processes. Markov chains X on countable state spaces can be chosen [CE10] to take values in the set of unit vectors $\{e_i : i \in \mathbb{N}\}$ of the sequence space $\ell^2 \subset \mathbb{R}^{\mathbb{N}}$, with jumps ΔX taking values $e_i - e_j$, $i, j \in \mathbb{N}$.*

5. *Note that in suitable cases, the pure jump martingale $\eta * \widetilde{\mu}$ ($\eta \in \mathcal{H}^2_\nu$) can be written as a series of mutually orthogonal martingales. More precisely, assume that the compensator coincides with the product measure $\lambda \otimes dt$, i.e. $\zeta = 1$. Let $(f^n)_{n \in \mathbb{N}}$ be an orthonormal basis of the separable Hilbert space $L^2(\lambda)$ with scalar product $\langle f, g \rangle := \int_E f(e) g(e) \, \lambda(de)$. Let $\eta_t = \sum_{n \in \mathbb{N}} \langle \eta_t, f^n \rangle f^n$ be the basis expansion of η_t for $\eta \in \mathcal{H}^2_\nu$, $t \in [0, T]$. Then it holds (in $\mathcal{M}^2$)*

$$\eta * \widetilde{\mu} = \sum_{n \in \mathbb{N}} \int_0^{\cdot} \langle \eta_t, f^n \rangle \int_E f^n(e) \, \widetilde{\mu}(dt, de) =: \sum_{n \in \mathbb{N}} \int_0^{\cdot} \alpha^n_t \, dL^n_t = \sum_{n \in \mathbb{N}} \alpha^n \cdot L^n, \tag{1.3}$$

for $\alpha^n_t := \langle \eta_t, f^n \rangle$ and $L^n := f^n * \widetilde{\mu}$. Indeed, setting $F^n_t := \sum_{k=1}^n \langle \eta_t, f^k \rangle f^k = \sum_{k=1}^n \alpha^k_t f^k$ one sees that $\| \sum_{k=1}^\infty |\alpha^k|^2 \|_{L^1(P \otimes dt)} \leq \|\eta\|^2_{\mathcal{H}^2_\nu} < \infty$. By dominated convergence

$$\begin{aligned} \|F^n - \eta\|^2_{\mathcal{H}^2_\nu} &= E\left(\int_0^T \!\! \int_E |F^n_t(e) - \eta_t(e)|^2 \lambda(de) \, dt \right) \\ &= E\left(\int_0^T \sum_{k=n+1}^\infty |\alpha^k_t|^2 \, dt \right) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Isometry implies that the stochastic integrals $F^n * \widetilde{\mu}$ converge to $\eta * \widetilde{\mu}$ in $\mathcal{M}^2$, proving (1.3).

In particular, we see how the PRP (1.2) with respect to a random measure can be rewritten as series of ordinary stochastic integrals with respect to scalar-valued strongly orthogonal *martingales L^n, which are Lévy processes with deterministic characteristics $(0, 0, \int f^n(e)\,\lambda(de))$. In this sense, the general condition (1.2) links well with results on PRP and BSDEs for Lévy processes in [NS00, NS01] who study a specific Teugels martingale basis consisting of compensated power jump processes for Lévy processes which satisfy exponential moment conditions. For a systematic analysis of related PRP results, comprising general Levy processes, see [DTE13, DTE15].*

6. *Note that the previous arguments extend to the general case with $\zeta \neq 1$ in (1.1), letting f^n be in $\mathcal{L}^2(\widetilde{\mu})$ such that for all $t \in [0,T]$ the sequence $(f_t^n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\lambda_t)$ for $d\lambda_t = \zeta_t d\lambda$ with scalar product $\langle f_t, g_t\rangle_t := \int_E f_t(e)g_t(e)\,\zeta_t(e)\lambda(de)$. Analogously to case 5. above, with $\alpha_t^n := \langle \eta_t, f_t^n\rangle_t$ and $L^n := f^n * \widetilde{\mu}$ one gets equalities of martingales (in $\mathcal{M}^2$)*

$$\eta * \widetilde{\mu} = \sum_{n\in\mathbb{N}} \int_0^{\cdot} \langle \eta_t, f_t^n\rangle_t \int_E f_t^n(e)\,\widetilde{\mu}(dt, de) =: \sum_{n\in\mathbb{N}} \alpha^n \cdot L^n\,. \tag{1.4}$$

To proceed, we now define a solution of the Backward SDE with jumps to be a triple $(Y_t, Z_t, U_t)_{0\le t\le T}$ of processes in the space $\mathcal{S}^p \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ for a suitable $p \in (1, \infty]$ that satisfies

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s)\,ds - \int_t^T Z_s\,dW_s - \int_t^T\!\!\int_E U_s(e)\,\widetilde{\mu}(ds, de), \quad 0 \le t \le T, \tag{1.5}$$

for given data (ξ, f), consisting of a $\mathcal{F}_T$-measurable random variable ξ and a generator function $f_t(y, z, u) = f(\omega, t, y, z, u)$. The values p will be specified below in the respective results, although a particular focus will be on bounded BSDE solutions (i.e. $p = \infty$). Because we permit ν to be time-inhomogeneous with a bounded but possibly non-constant density ζ in (1.1), it does not hold in general that U_t is a.e. in $L^2(\lambda)$ for $U \in \mathcal{H}^2_\nu$. This technical complications: One needs to define the generator function f in the U-component on a suitable space larger than $L^2(\lambda)$ that is still fairly accessible, while being still clear in product-measurability. A sufficiently large space may well require that f is permitted to take possibly non-finite values. To this end, we denote by $L^0(\mathcal{B}(E), \lambda)$ the space of all $\mathcal{B}(E)$-measurable functions with the topology of convergence in measure and we define

$$|u - u'|_t := \Big(\int_E |u(e) - u'(e)|^2\,\zeta_t(e)\,\lambda(de)\Big)^{\frac{1}{2}}, \tag{1.6}$$

for any functions u, u' in $L^0(\mathcal{B}(E), \lambda)$. Terminal conditions ξ for BSDE considered in this chapter will be taken to be square integrable $\xi \in L^2(\mathcal{F}_T)$ and often even as bounded $\xi \in L^\infty(\mathcal{F}_T)$. Generator functions $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^0(\mathcal{B}(E), \lambda) \to \overline{\mathbb{R}}$ are always taken to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(L^0(\mathcal{B}(E), \lambda))$-measurable. Main results such as Theorems 1.11, 1.18 and 1.28 will be stated for families of generators having the form

$$f_t(y, z, u) := \widehat{f}_t(y, z) + \int_A g_t(y, z, u(e), e)\zeta_t(e)\lambda(de) \quad \text{(where finitely defined)} \tag{1.7}$$

and $f_t(y,z,u) := \infty$ elsewhere, or more specially (for a g-component not depending on y,z)

$$f_t(y,z,u) := \widehat{f}_t(y,z) + \int_A g_t(u(e),e)\,\zeta_t(e)\,\lambda(de) \quad \text{(where finitely defined)} \tag{1.8}$$

and $f_t(y,z,u) := \infty$ elsewhere, for a set A in $\mathcal{B}(E)$ and component functions $\widehat{f}$, g where $\widehat{f} : \Omega \times [0,T] \times \mathbb{R}^{1+d} \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1})$-measurable and $g : \Omega \times [0,T] \times \mathbb{R}^{2+d} \times E \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+2}) \otimes \mathcal{B}(E)$-measurable. Clearly statements for generators of the form (1.7) are also true for those of the form (1.8), since the latter is a special type of the former. (In)finite activity relates to generators with $\lambda(A) < \infty$ (respectively $\lambda(A) = \infty$). A simple but useful technical Lemma clarifies how we can (and always will) choose a bounded representative for U in a BSDE solution (Y,Z,U) with bounded Y.

Lemma 1.2. *Let $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ be a solution of some JBSDE (1.5) with data (ξ,f). Then there exists a representative U' of U, bounded pointwise by $2|Y|_\infty$, such that $U' = U$ in $\mathcal{H}^2_\nu$ and $P \otimes dt$-a-e., and (Y,Z,U') solves the BSDE (ξ,f).*

Proof. The argument is short and worth stating in our general setting, although the idea is similar to e.g. in [Mor09, Corollary 1] or [Bec06](proof of Theorem 3.5). Use that $\mu(\omega;dt,de) = \sum_{s\geq 0} 1_D(\omega,s)\delta_{(s,\beta_s(\omega))}(dt,de)$ for an optional E-valued process β and a thin set D, since μ is an integer-valued random measure ([JS03],II.§1b). Clearly $\Delta Y_t(\omega) = (Y_t - Y_{t-})(\omega) = \int_E U_t(\omega,e)\,\mu(\omega;\{t\},de)$ is equal to $1_D(\omega,t)U_t(\omega,\beta_t(\omega))$ and bounded by $2|Y|_\infty$. Moreover for $U'_t(\omega,e) := U_t(\omega,e)\mathbb{1}_D(\omega,t)\mathbb{1}_{\beta_t}(e)$, it clearly holds $U_t(\omega,\beta_t(\omega)) = U'_t(\omega,\beta_t(\omega))$ on D, and $\sum_{s\geq 0} 1_D(\omega,s)|U_s - U'_s|^2(\omega,\beta_s(\omega)) = 0$ implies $E[|U-U'|^2 * \nu_T] = E[|U-U'|^2 * \mu_T] = 0$. Since $U = U'$ in $\mathcal{H}^2_\nu$ and $U_t = U'_t$ in $L^0(\mathcal{B}(E),\lambda)$, the BSDE is solved by (Y,Z,U'). □

Under these conditions, we can and will take U to be bounded by twice the supremum norm of Y; recalling $|U|_\infty := \text{ess sup}_{(t,e)} |U_t(e)|$ for $U \in \mathcal{H}^2_\nu$ yields $|U|_\infty \leq 2|Y|_\infty$ for any bounded BSDE solution (Y,Z,U). Next, we show that the stochastic integrals of bounded BSDE solutions are BMO-martingales whenever some truncated generator function is bounded from below by $-\langle M\rangle$ or from above by $\langle M\rangle$, for a BMO-martingale M. Moreover, their BMO-norms depend only on $|Y|_\infty$, the BMO-norm of M and the time horizon T.

Lemma 1.3. *Let $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ be a bounded solution to the BSDE (ξ,f). Assume there is a BMO-martingale $M \in BMO(P)$ such that $\int_t^T f_s(Y_{s-},Z_s,U_s)\,ds \leq \langle M\rangle_T - \langle M\rangle_t$ or $-\int_t^T f_s(Y_{s-},Z_s,U_s)\,ds \leq \langle M\rangle_T - \langle M\rangle_t$. Then $\int Z\,dW$ and $U * \widetilde{\mu}$ are BMO-martingales and their BMO-norms (resp. L^2-norms) are bounded by a constant depending on $|Y|_\infty$ and $\|M\|_{BMO(P)}$ (resp. on $|Y|_\infty$, $\|M\|_{\mathcal{M}^2}$).*

Proof. Applying Itô's formula for general semimartingales to $\exp(\pm Y_t)$ yields

$$\begin{aligned}\exp(\pm Y_t) = \exp(\pm\xi) \mp \int_t^T &\exp(\pm Y_{s-}) Z_s\, dW_s \\ &\pm \int_t^T \exp(\pm Y_{s-}) f_s(Y_{s-}, Z_s, U_s)\, ds - \frac{1}{2}\int_t^T \exp(\pm Y_{s-})\|Z_s\|^2\, ds \\ &- \int_t^T\int_E \exp(\pm Y_{s-})(\exp(\pm U_s(e)) - 1)\, \widetilde{\mu}(ds, de) \\ &- \int_t^T\int_E \exp(\pm Y_{s-})(\exp(\pm U_s(e)) - 1 \mp U_s(e))\, \nu(ds, de)\end{aligned}$$

Taking conditional expectations and noting by the first order Taylor expansion that it holds $\exp(\pm U_s(e)) - 1 \mp U_s(e) = \frac{1}{2}\exp(\pm V_s(e))|U_s(e)|^2$ for some $V_s(e)$ between 0 and $U_s(e)$, we obtain the estimate

$$\begin{aligned}\frac{1}{2}&\exp(-c)E_t\left[\int_t^T \|Z_s\|^2\, ds\right] + \frac{1}{2}\exp(-3c)E_t\left[\int_t^T\int_E |U_s(e)|^2\, \nu(ds, de)\right] \\ &\le E_t\left[\int_t^T \exp(\pm Y_{s-})(\frac{1}{2}\|Z_s\|^2 + \int_E (\exp(\pm U_s(e)) - 1 \mp U_s(e))\, \zeta_s(e)\, \lambda(de))\, ds\right] \\ &\le E_t\left[\exp(\pm\xi) - \exp(\pm Y_t) \pm \int_t^T \exp(\pm Y_{s-}) f_s(Y_{s-}, Z_s, U_s)\, ds\right] \\ &\le e^c - e^{-c} + e^c\|M\|_{BMO(P)}.\end{aligned}$$

On the last line we used the assumption on the $BMO(P)$-martingale M. Since the jumps of $U * \widetilde{\mu}$ are the jumps of Y, hence bounded, the claim follows. □

1.2 Comparison theorems and a-priori-estimates

The next proposition states a result that provides the basis for the main comparison Theorem 1.11 and the a-priori-estimate Theorem 1.13 of this section. As usual for BSDE comparison results, the proof relies on a linearization technique and a change of measure argument. In a framework with random measures, it is very close to the seminal Theorem 2.5 from [Roy06] with some slight generalization that are needed in the sequel. Yet, we can same line of proof with (slightly) more general assumptions. Some details of the change of measure argument are elaborated slightly differently (cf. end of the proof) and we assume less on the generators. Instead of imposing specific conditions on the generators which imply existence of solutions, we only insist that we have solutions and impose a generalized (A_γ)-condition as explained in Example 1.10.1.

Proposition 1.4. *Let $(Y^i, Z^i, U^i) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ be solutions to the BSDE (1.5) for data (ξ_i, f_i), $i = 1, 2$. Assume that the generator f_2 is Lipschitz continuous with respect to y*

and z. Let $\gamma : \Omega \times [0,T] \times \mathbb{R}^{d+3} \times E \to [-1,\infty)$ with $(\omega, t, y, z, u, u', e) \mapsto \gamma_t^{y,z,u,u'}(e)$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable function such that for $\overline{\gamma} := \gamma^{Y^2_-, Z^2, U^1, U^2}$ it holds

$$f_2(t, Y^2_{t-}, Z^2_t, U^1_t) - f_2(t, Y^2_{t-}, Z^2_t, U^2_t) \le \int_E \overline{\gamma}_t(e)\,(U^1_t(e) - U^2_t(e))\,\zeta_t(e)\,\lambda(de),\ P \otimes dt\text{-a.s.}$$

$$\text{and } \mathcal{E}\Big(\int \beta\, dW + \overline{\gamma} * \widetilde{\mu}\Big) \text{ is a martingale for } \beta \text{ from (1.10).} \tag{1.9}$$

Then a comparison result holds, that is $\xi_1 \le \xi_2$ and $f_1(t, Y^1_{t-}, Z^1_t, U^1_t) \le f_2(t, Y^1_{t-}, Z^1_t, U^1_t)$ together imply $Y^1_t \le Y^2_t$ for all $t \le T$.

Proof. We define $\widehat{\xi} := \xi_1 - \xi_2$, $\widehat{Y} := Y^1 - Y^2$, $\widehat{Z} := Z^1 - Z^2$ and $\widehat{U} := U^1 - U^2$. The processes

$$\begin{aligned}
\alpha_s &:= 1_{\{Y^1_{s-} \neq Y^2_{s-}\}} \frac{f_2(s, Y^1_{s-}, Z^1_s, U^1_s) - f_2(s, Y^2_{s-}, Z^1_s, U^1_s)}{(Y^1_{s-} - Y^2_{s-})}\,, \\
\beta_s &:= 1_{\{Z^1_s \neq Z^2_s\}} \frac{f_2(s, Y^2_{s-}, Z^1_s, U^1_s) - f_2(s, Y^2_{s-}, Z^2_s, U^1_s)}{\|Z^1_s - Z^2_s\|^2}(Z^1_s - Z^2_s)
\end{aligned} \tag{1.10}$$

and $R_t := \exp(\int_0^t \alpha_s\, ds)$ are bounded due to the Lipschitz assumption on f_2. As in [Roy06], applying Itô's formula to $R\widehat{Y}$ between $\tau \wedge t$ and $\tau \wedge T$ for some stopping times τ yields

$$\begin{aligned}
(R\widehat{Y})_{\tau\wedge t} = (R\widehat{Y})_{\tau\wedge T} &+ \int_{\tau\wedge t}^{\tau\wedge T} R_s(f_1(s, Y^1_{s-}, Z^1_s, U^1_s) - f_2(s, Y^2_{s-}, Z^2_s, U^2_s))\, ds \\
&- \int_{\tau\wedge t}^{\tau\wedge T} R_s \widehat{Z}_s\, dW_s - \int_{\tau\wedge t}^{\tau\wedge T}\int_E R_s \widehat{U}_s(e)\,\widetilde{\mu}(ds, de) - \int_{\tau\wedge t}^{\tau\wedge T} R_s \alpha_s \widehat{Y}_{s-}\, ds.
\end{aligned}$$

Set $M := \int R\widehat{Z}\, dW + (R\widehat{U}) * \widetilde{\mu}$ and $N := \int \beta\, dW + \overline{\gamma} * \widetilde{\mu}$. Then $dQ := \mathcal{E}(N)_T dP$ defines an absolutely continuous probability measure Q due to the martingale property of the stochastic exponential $\mathcal{E}(N) \ge 0$ (see [HWY92], Lemma $9.40.$). Moreover, by Girsanov's theorem $L := M - \langle M, N\rangle$ is a local Q-martingale, and

$$f_1(s, Y^1_{s-}, Z^1_s, U^1_s) - f_2(s, Y^2_{s-}, Z^2_s, U^2_s) \le \alpha_s \widehat{Y}_{s-} + \beta_s \widehat{Z}_s + \int_E \overline{\gamma}_s(e) \widehat{U}_s(e)\,\zeta_s(e)\,\lambda(de)$$

holding $P \otimes ds$-a.e. implies

$$(R\widehat{Y})_{\tau\wedge t} \le (R\widehat{Y})_{\tau\wedge T} - (L^\tau_T - L^\tau_t). \tag{1.11}$$

Localizing L along a sequence of stopping times $\tau_n \uparrow \infty$ and taking conditional expectations, we obtain $E^Q_t[(R\widehat{Y})_{t\wedge\tau^n}] \le E^Q_t[(R\widehat{Y})_{\tau^n\wedge T}]$ for each $n \in \mathbb{N}$. Dominated convergence yields $R_t\widehat{Y}_t \le E^Q_t[R_T\widehat{\xi}] \le 0$ and thus $Y^1_t \le Y^2_t$. □

Remark 1.5. *Switching the roles of f_1 and f_2 one can show that if f_1 is Lipschitz in y and z and satisfies (1.9) instead of f_2, then $\xi_1 \le \xi_2$, $f_1(t, Y^2_{t-}, Z^2_t, U^2_t) \le f_2(t, Y^2_{t-}, Z^2_t, U^2_t)$, $P \otimes dt$-a.s. together imply $Y^1_t \le Y^2_t$.*

Example 1.6. *Sufficient conditions for $\mathcal{E}(\overline{\gamma} * \widetilde{\mu})$ to be a martingale are, for instance*

1. $\Delta(\overline{\gamma} * \widetilde{\mu}) > -1$ *and* $E(\exp(\langle \overline{\gamma} * \widetilde{\mu} \rangle_T)) = E\left(\exp\left(\int_0^T \int_E |\overline{\gamma}_s(e)|^2 \, \nu(ds, de)\right)\right) < \infty$ *(see Theorem 9, [PS08]). In particular, this holds if* $\int_E |\overline{\gamma}_s(e)|^2 \, \zeta_s(e) \, \lambda(de) < const. < \infty$ $P \otimes ds$*-a.e. and* $\overline{\gamma} > -1$.
2. $\Delta(\overline{\gamma} * \widetilde{\mu}) \geq -1 + \delta$ *for some* $\delta > 0$ *and* $\overline{\gamma} * \widetilde{\mu}$ *is a* $BMO(P)$*-martingale. This is due to Kazamaki [Kaz79].*
3. $\Delta(\overline{\gamma} * \widetilde{\mu}) \geq -1$ *and* $\overline{\gamma} * \widetilde{\mu}$ *is a uniformly integrable martingale and* $E(\exp(\langle \overline{\gamma} * \widetilde{\mu} \rangle_T)) < \infty$ *(see Theorem I.8, [LM78]). Such a condition is satisfied when* $\overline{\gamma}$ *is bounded and* $|\overline{\gamma}| \leq \psi, \; P \otimes dt \otimes d\lambda$*-a.s. for a function* $\psi \in L^2(\lambda)$ *and* $\zeta \equiv 1$*. The latter is what is required for instance in the comparison Theorem 4.2 of [QS13].*

Note that under above conditions, also the stochastic exponential $\mathcal{E}(\int \beta dW + \overline{\gamma} * \widetilde{\mu})$ *for* β *bounded and predictable is a martingale, as it is easily seen by Novikov's criterion.*

In the statement of Proposition 1.4, the dependence of the process $\overline{\gamma}$ on the BSDE solutions is not needed for the proof as the same result holds if $\overline{\gamma}$ is just a predictable process such that the estimate on the generator f_2 and the martingale property (1.9) hold. The further functional dependence is needed for the sequel, as required in the following

Definition 1.7. *An* $\overline{\mathbb{R}}$*-valued generator function* f *is said to satisfy condition* $(\mathbf{A}_\gamma)$ *if there is a* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$*-measurable function* $\gamma : \Omega \times [0, T] \times \mathbb{R}^{d+3} \times E \to (-1, \infty)$ *given by* $(\omega, t, y, z, u, u', e) \mapsto \gamma_t^{y,z,u,u'}(e)$ *such that for all* $(Y, Z, U, U') \in \mathcal{S}^\infty \times \mathcal{H}^2 \times (\mathcal{H}^2_\nu)^2$ *with* $|U|_\infty < \infty$, $|U'|_\infty < \infty$ *it holds for* $\overline{\gamma} := \gamma^{Y_-, Z, U, U'}$

$$f_t(Y_{t-}, Z_t, U_t) - f_t(Y_{t-}, Z_t, U'_t) \leq \int_E \overline{\gamma}_t(e)(U_t(e) - U'_t(e))\zeta_t(e)\lambda(de), \; P \otimes dt\text{-a.s.} \tag{1.12}$$

and $\mathcal{E}(\int \beta dW + \overline{\gamma} * \widetilde{\mu})$ *is a martingale for every bounded and predictable* β.

A function f *satisfies condition* $(\mathbf{A}'_\gamma)$ *if the above holds for all bounded* U *and* U' *with additional property that* $U * \widetilde{\mu} \in BMO(P)$ *and* $U' * \widetilde{\mu} \in BMO(P)$.

Clearly, existence and applicability of a suitable comparison result of solutions to JBSDEs implies their uniqueness. In other words assuming there exists a bounded solution for a Lipschitz driver with respect to y and z which satisfies (A_γ) or (A'_γ), we obtain that such a solution is unique.

Example 1.8. *A natural candidate* γ *for drivers* f *of the form* (1.7) *is given by* $\gamma_s^{y,z,u,u'}(e) := \int_0^1 \frac{\partial}{\partial u} g_s(y, z, tu + (1-t)u', e) \, dt \, \mathbb{1}_A(e)$*, assuming differentiability of* g*. Indeed, we have*

$$\begin{aligned}
\gamma_s^{y,z,u,u'}(e)(u - u') &= \int_0^1 \frac{\partial}{\partial t} \left[(g_s(y, z, tu + (1-t)u', e)) \right] \, dt \, \mathbb{1}_A(e) \\
&= (g_s(y, z, u, e) - g_s(y, z, u', e))\mathbb{1}_A(e),
\end{aligned}$$

and hence the function γ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable since

$$\gamma_s^{y,z,u,u'}(e) = \begin{cases} \frac{g_s(y,z,u,e)-g_s(y,z,u',e)}{u-u'} \mathbb{1}_A(e), & u \neq u' \\ 0, & u = u', \end{cases}$$

and $g, \frac{\partial g}{\partial u}$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable. By the mean value theorem γ has the form

$$\gamma_s^{y,z,u,u'}(e) = \frac{\partial}{\partial u} g(s,y,z,v,e) \mathbb{1}_A(e), \tag{1.13}$$

for some v between u and u'. For generators of the form (1.8) γ simplifies to

$$\gamma_s^{y,z,u,u'}(e) = \int_0^1 \frac{\partial}{\partial u} g_s(tu + (1-t)u', e)\, dt\, \mathbb{1}_A(e)\,.$$

Definition 1.9. *A generator f satisfies condition (A_{fin}) (on D or for elements in D) or (A_{infi}) if*

1. *(A_{fin}): f is of the form (1.7) with $\lambda(A) < \infty$, is Lipschitz continuous with respect to y and z, and the map $u \mapsto g(t,y,z,u,e)$ is continuously differentiable for all (ω,t,y,z,e) (in D) such that the derivative is strictly greater than -1 (on $D \subseteq \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times E$) and locally bounded (in u) from above, uniformly in (ω,t,y,z,e).*

2. *(A_{infi}): f is of the form (1.8), is Lipschitz continuous with respect to y and z, and the map $u \mapsto g_t(u,e)$ is twice continuously differentiable for all (ω,t,e) with the derivatives being locally (in u) bounded uniformly in (ω,t,e), the first derivative bounded away from -1 with a lower bound $-1+\delta$ for some $\delta > 0$, and $\frac{\partial g}{\partial u}(t,0,e) \equiv 0$.*

Example 1.10. *Sufficient conditions for (A_γ) and (A'_γ) are*

1. *γ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable satisfying the inequality in (1.12) and*

   $$C_1(1 \wedge |e|) \leq \gamma_t^{y,z,u,u'}(e) \leq C_2(1 \wedge |e|),$$

   for some $C_1 \in (-1,0]$ and $C_2 > 0$ on $E = \mathbb{R}^d \backslash \{0\}$. In this case $\exp\left(\langle \int \beta dW + \overline{\gamma} * \widetilde{\mu}\rangle_T\right)$ is clearly bounded and the jumps of $\int \beta dW + \overline{\gamma} * \widetilde{\mu}$ are bigger than -1. Hence $\mathcal{E}\left(\int \beta dW + \overline{\gamma} * \widetilde{\mu}\right)$ is a positive martingale ([PS08], Theorem 9). Thus Definition 1.7 generalizes the original (A_γ)-condition introduced in [Roy06] for Poisson random measures.

2. *(A_{fin}) is sufficient for (A_γ). This follows from Example 1.6.1, (1.13) and $\lambda(A) < \infty$.*

3. *(A_{infi}) is sufficient for (A'_γ). To see this, let u, u' be bounded by c and γ be the natural candidate in Example 1.8. By the mean value theorem there exist $v(e)$ between u and u'*

and $\widetilde{v}(e)$ between 0 and $v(e)$ such that

$$\begin{aligned}\gamma_s^{y,z,u,u'}(e) &= \frac{\partial}{\partial u} g(s, v(e), e) \mathbb{1}_A(e) \\ &= \Big(\frac{\partial}{\partial u} g(s, v(e), e) - \frac{\partial}{\partial u} g(s, 0, e)\Big) \mathbb{1}_A(e) = v(e) \frac{\partial^2}{\partial u^2} g(s, \widetilde{v}(e), e) \mathbb{1}_A(e).\end{aligned}$$

So $\gamma_s^{y,z,u,u'}(e)$ is bounded uniformly in (ω, s, e) by

$$|\gamma_s^{y,z,u,u'}(e)| \leq \sup_{|u|\leq c} \Big|\frac{\partial^2}{\partial u^2} g(s, y, z, u, e)\Big|(|u| + |u'|),$$

hence $\int \beta dW + \overline{\gamma} * \widetilde{\mu}$ is a BMO-martingale by the BMO-property of $U * \widetilde{\mu}$ and $U' * \widetilde{\mu}$ with some lower bound $-1 + \delta$ for its jumps. And $\mathcal{E}(\int \beta dW + \overline{\gamma} * \widetilde{\mu})$ is a martingale by Kazamaki's criterion of Example 1.6.

As an application of the above, we can now provide simple conditions for comparison in terms of concrete properties of the generator function, which are much easier to verify than the more general but abstract conditions on the existence of a suitable function γ as in Proposition 1.4 or the general conditions by [CE10]. Note that no convexity is required in the z or u argument of the generator. The result will be applied later to prove existence and uniqueness of JBSDE solutions.

Theorem 1.11 (Comparison Theorem)**.** *A comparison result between bounded BSDE solutions in the sense of Proposition 1.4 holds true in each of the following cases:*

1. (finite activity) *f_2 satisfies (A_{fin}).*
2. (infinite activity) *f_2 satisfies (A_{infi}) and $U^1 * \widetilde{\mu}$ and $U^2 * \widetilde{\mu}$ are BMO(P)-martingales for the corresponding JBSDE solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2).*

Proof. This follows directly from Proposition 1.4 and Example 1.10, noting that representation (1.13) in connection with condition (A_{fin}) resp. (A_{infi}) meets the sufficient conditions in Example 1.6. □

Unlike classical a-priori estimates that offer some L^2-norm estimates for the BSDE solution in terms of the data, the next result gives a simple L^∞-estimate for the Y-component of the solution. Such will be useful for the derivation of BSDE solution bounds and for truncation arguments.

Proposition 1.12. *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ be a solution to the BSDE (ξ, f) with $\xi \in L^\infty(\mathcal{F}_T)$. Let f be Lipschitz continuous with respect to (y, z) with Lipschitz constant $K_f^{y,z}$ and satisfying (A_γ) with $f_\cdot(0, 0, 0)$ bounded. Then*

$$|Y_t| \leq \exp\big(K_f^{y,z}(T - t)\big)\big(|\xi|_\infty + (T - t)|f_\cdot(0, 0, 0)|_\infty\big)$$

holds for all $t \leq T$.

Proof. Set $(Y^1, Z^1, U^1) = (Y, Z, U)$, $(\xi^1, f^1) = (\xi, f)$, $(Y^2, Z^2, U^2) = (0, 0, 0)$ and $(\xi^2, f^2) = (0, f)$, Then following the proof of Proposition 1.4, equation (1.11) becomes

$$(RY)_{\tau\wedge t} \leq (RY)_{\tau\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s f_s(0,0,0)\, ds - (L_T^\tau - L_t^\tau), \quad t \in [0, T],$$

for all stopping times τ where $L := M - \langle M, N\rangle$ is in $\mathcal{M}_{loc}(Q)$, $M := \int RZ\, dW + (RU) * \widetilde{\mu}$ is in $\mathcal{M}^2$, $N := \int \beta\, dW + \overline{\gamma} * \widetilde{\mu}$ with $\overline{\gamma} := \gamma^{0,0,U,0}$ and the probability measure $Q \sim P$ is given by $dQ := \mathcal{E}(N)_T dP$. Localizing $(L_t)_{0\leq t\leq T}$ along some sequence $(\tau^n)_{n\in\mathbb{N}} \uparrow \infty$ yields $E_t^Q[(RY)_{\tau^n\wedge t}] \leq E_t^Q[(RY)_{\tau^n\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s f_s(0,0,0)\, ds]$. By dominated convergence, we conclude that P-a.e

$$Y_t \leq E_t^Q\left[\frac{R_T}{R_t}\xi + \int_t^T \frac{R_s}{R_t} f_s(0,0,0)\, ds\right] \leq e^{K_f^{y,z}(T-t)}(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty).$$

Analogously, if we define $\overline{N} := \int \beta\, dW + \overline{\overline{\gamma}} * \widetilde{\mu}$ with $\overline{\overline{\gamma}} := \gamma^{0,0,0,U}$, and $\overline{Q}$ equivalent to P via $d\overline{Q} := \mathcal{E}(\overline{N})_T dP$, we deduce that $\overline{L} := M - \langle M, \overline{N}\rangle \in \mathcal{M}_{loc}(\overline{Q})$ and

$$(RY)_{\tau\wedge t} \geq (RY)_{\tau\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s f_s(0,0,0)\, ds - (\overline{L}_T^\tau - \overline{L}_t^\tau), \quad t \in [0, T],$$

for all stopping times τ. This yields the lower bound. □

Again, we can specify explicit conditions on the generator function that are sufficient to ensure the more abstract assumptions of the previous result.

Theorem 1.13. *Let* $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}_\nu^2$ *be a solution to the BSDE* (ξ, f) *with* ξ *in* $L^\infty(\mathcal{F}_T)$, f *being Lipschitz continuous with respect to* y *and* z *with Lipschitz constant* $K_f^{y,z}$ *such that* $f_\cdot(0,0,0)$ *is bounded. Assume that one of the following conditions holds:*

1. (finite activity) f *has property* (A_{fin}).
2. (infinite activity) f *has property* (A_{infi}) *and* $U * \widetilde{\mu}$ *is a BMO(*P*)-martingale.*

Then

$$|Y_t| \leq \exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty)$$

holds for all $t \leq T$, *and in particular* $|Y|_\infty \leq \exp(K_f^{y,z} T)(|\xi|_\infty + T|f_\cdot(0,0,0)|_\infty)$.

Proof. This follows directly from Proposition 1.12 and Example 1.10, since f satisfies condition (A_γ) (resp. (A'_γ)) using equation (1.13). □

In the last part of this section we apply our comparison theorem for more concrete generators. To this end, we consider a truncation $\widetilde{f}$ of a generator f at truncation bounds $a < b$ (depending on time only), given by

$$\widetilde{f}_t(y,z,u) := f_t(\kappa(t,y), z, \kappa(t,y+u) - \kappa(t,y)), \tag{1.14}$$

with $\kappa(t,y) := (a(t) \vee y) \wedge b(t)$. Next, we show that if a generator satisfies (A_γ) within the truncation bounds, then the truncated generator satisfies (A_γ) everywhere.

Lemma 1.14. *Let f satisfy (1.12) for Y, U such that*

$$a(t) \leq Y_{t-}, Y_{t-} + U_t(e), Y_{t-} + U'_t(e) \leq b(t), \quad t \in [0,T],$$

and let γ satisfy one of the conditions of Example 1.6 for the martingale property of $\mathcal{E}(\overline{\gamma} * \widetilde{\mu})$. Then $\widetilde{f}$ satisfies (1.12). Especially, if f satisfies (A_{fin}) on the set where $a(t) \leq y, y+u \leq b(t)$ then $\widetilde{f}$ is Lipschitz in y and z, locally Lipschitz in u and satisfies (A_γ).

Proof. Indeed, we have

$$\begin{aligned}
&\widetilde{f}_t(Y_{t-}, Z_t, U_t) - \widetilde{f}_t(Y_{t-}, Z_t, U'_t) \\
&= f_t(\kappa(t,Y_{t-}), Z_t, \kappa(t, Y_{t-} + U_t) - \kappa(t,Y_{t-})) - f_t(\kappa(t,Y_{t-}), Z_t, \kappa(t, Y_{t-} + U'_t) - \kappa(t,Y_{t-})) \\
&\leq \int_E \overline{\gamma}_t(e)(\kappa(t, Y_{t-} + U_t(e)) - \kappa(t, Y_{t-} + U'_t(e)))\, \zeta_t(e)\, \lambda(de) \\
&\leq \int_E \overline{\gamma}_t(e)(\mathbb{1}_{\{\overline{\gamma} \geq 0, U \geq U'\}} + \mathbb{1}_{\{\overline{\gamma} < 0, U < U'\}})(U_t(e) - U'_t(e))\, \zeta_t(e)\, \lambda(de),
\end{aligned}$$

due to the monotonicity of $x \mapsto \kappa(t,x)$. Setting $\overline{\gamma^*} := \overline{\gamma}(\mathbb{1}_{\{\overline{\gamma} \geq 0, U \geq U'\}} + \mathbb{1}_{\{\overline{\gamma} < 0, U < U'\}})$ we see that the stochastic exponential $\mathcal{E}(\int \beta dW + \overline{\gamma^*} * \widetilde{\mu})$ is a martingale for all bounded and predictable processes β and $\widetilde{f}$ satisfies (1.12). The latter claim easily follows from the fact that if f satisfies (A_{fin}) on $a(t) \leq y, y+u \leq b(t)$ then using Example 1.10.2. f satisfies (1.12) on the set of (t,ω) such that $a(t) \leq Y_{t-}, Y_{t-} + U_t(e), Y_{t-} + U'_t(e) \leq b(t)$. The Lipschitz properties of $\widetilde{f}$ follow from the fact that κ is a contraction and f is Lipschitz within the truncation bounds. ☐

We are now able to give concrete estimates for bounded solutions to a BSDE (ξ, f), with generator component $\widehat{f}$ being linearly bounded in y.

Proposition 1.15. *Let f be a generator of the form (1.7) with $|\widehat{f}_t(y,z)| \leq K_1 + K_2|y|$ for some $K_1, K_2 \geq 0$, $g_t(y,z,0,e) \equiv 0$ and $\xi \in L^\infty(\mathcal{F}_T)$ with $c_1 \leq \xi \leq c_2$ for some $c_1, c_2 \in \mathbb{R}$. Assume that there are solutions a and b to the ODEs $y'(t) = K_1 + K_2|y(t)|$, $y(T) = c_1$ and $y'(t) = -(K_1 + K_2|y(t)|)$, $y(T) = c_2$ respectively, such that $a \leq b$ on $[0,T]$. If the truncated generator $\widetilde{f}$ in (1.14) satisfies (A_γ) and is Lipschitz in y and z, then any solution $(\widetilde{Y}, \widetilde{Z}, \widetilde{U}) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the JBSDE $(\xi, \widetilde{f})$ also solves the JBSDE (ξ, f) and satisfies $a(t) \leq \widetilde{Y}_t \leq b(t)$, $t \in [0,T]$.*

Proof. We set $Y_t := \kappa(t, \widetilde{Y}_t)$, $Z_t := \widetilde{Z}_t$, $U_t(e) := \kappa(t, \widetilde{Y}_{t-} + \widetilde{U}_t(e)) - \kappa(t, \widetilde{Y}_{t-})$ and

$$f_t^i(y,z,u) := \widehat{f}_t^i(\kappa(t,y),z) + \int_E g_t(\kappa(t,y),z,\kappa(t,y+u)-\kappa(t,y),e)\,\zeta_t(e)\,\lambda(de)$$

with $\widehat{f}_t^1(y,z) := -(K_1 + K_2|y|)$, $\widehat{f}_t^2(y,z) := \widehat{f}_t(y,z)$ and $\widehat{f}_t^3(y,z) := K_1 + K_2|y|$. By the assumptions on the ODEs, we have that $(a(t),0,0)$ solves the BSDE (c_1, f^1) and $(b(t),0,0)$ solves the BSDE (c_2, f^3). Taking into account that $\widetilde{f}^1 \leq \widetilde{f}^2 \leq \widetilde{f}^3$, $c_1 \leq \xi \leq c_2$ and $\widetilde{f}^2$ satisfies (A_γ), comparison theorem Proposition 1.4 yields $a(t) \leq \widetilde{Y}_t \leq b(t)$. Hence, Y and $\widetilde{Y}$ are indistinguishable, $U = \widetilde{U}$ in $\mathcal{H}^2_\nu$ and $(\widetilde{Y}, \widetilde{Z}, \widetilde{U})$ solves the BSDE (ξ, f). □

In the next section, we apply these results to two situations, namely using Corollary 1.19 to give an alternative proof of [Bec06, Theorem 3.5] via a comparison principle instead of an argument with stopping times. Moreover, the estimates in Corollary 1.21 are applied to solve the power utility maximization problem via a JBSDE approach in section 1.4.2.

1.3 Existence and Uniqueness of bounded solutions

This sections provided the results on wellposedness for BSDE with jumps, upon which the optimal control applications of section 1.4 do rely.

1.3.1 The case of finite activity

Definition 1.16. *A generator function satisfies (B_γ) if f is Lipschitz continuous in y and z, locally Lipschitz continuous in u, $f_\cdot(0,0,0)$ is bounded and f satisfies (A_γ).*

We now can show the following existence and uniqueness result which will be applied in Theorem 1.18, for A such that $\lambda(A) < \infty$.

Proposition 1.17. *Let $\xi \in L^\infty(\mathcal{F}_T)$ and f satisfies (B_γ). Then there exists a unique solution (Y,Z,U) in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f). Moreover for all $t \in [0,T]$, $|Y_t|$ is bounded by $\exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty)$.*

Proof. Consider the Lipschitz generator $f_t^c(y,z,u) := f_t(y,z,(u \vee (-c)) \wedge c)$ with $c > 0$ and Lipschitz constant K_{f^c}. By [Bec06, Propositions 3.2 and 3.3], there exists a unique solution $(Y^c, Z^c, U^c) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f^c) and it satisfies

$$|Y_t^c| \leq C E_t \left[|\xi|^2 + \int_t^T |f_s^c(0,0,0)|^2\,ds \right] \leq C(|\xi|^2_\infty + T|f_\cdot(0,0,0)|^2_\infty) < \infty,$$

for some constant $C = C(T, K_{f^c})$. Now Proposition 1.12 implies

$$|Y_t^c| \leq \exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty) \text{ for all } c > 0.$$

Choosing $c \geq 2\exp(K_f^{y,z}T)(|\xi|_\infty + T|f_\cdot(0,0,0)|_\infty)$ we get that (Y^c, Z^c, U^c) with $Y^c \in \mathcal{S}^\infty$ solves the BSDE (ξ, f) since U^c is bounded by c. Uniqueness follows by comparison. □

Now, we can apply this to the following setting

Theorem 1.18. *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let f satisfy (A_{fin}) with $f_\cdot(0,0,0)$ bounded. Then there exists a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f). Moreover for all $t \in [0,T]$, $|Y_t|$ is bounded by $\exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty)$.*

Proof. Noting that local Lipschitz continuity in u follows from the continuous differentiability of g in u with locally bounded derivative, this claim follows from a combination of Theorem 1.13 and Proposition 1.17. □

Corollary 1.19. *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let f be a generator satisfying (A_{fin}), with $g_t(y,z,0,e) \equiv 0$ and $|\widehat{f}_t(y,z)| \leq K_1 + K_2|y|$ for some $K_1, K_2 \geq 0$. Set*

$$b(t) = \begin{cases} (|\xi|_\infty + \frac{K_1}{K_2}) \exp\left(K_2(T-t)\right) - \frac{K_1}{K_2}, & K_2 \neq 0 \\ |\xi|_\infty + K_1(T-t), & K_2 = 0. \end{cases}$$

Then there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f) and moreover $|Y_t| \leq b_t$ for $t \in [0,T]$. Finally $\int Z\,dW$ and $U * \widetilde{\mu}$ are BMO(P)-martingales.

Proof. By Lemma 1.14 and Theorem 1.18, there is a unique solution (Y, Z, U) in the space $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE $(\xi, \widetilde{f})$. By Proposition 1.15, it also solves the BSDE (ξ, f) and $-b(t) \leq Y_t \leq b(t),\ \forall t \in [0,T]$. Uniqueness follows from the fact that one can apply the comparison Theorem 1.11 for generators satisfying (A_{fin}). The BMO property follows from Lemma 1.3. □

Remark 1.20. *1. While the statement of Corollary 1.19 is similar to [Bec06, Theorem 3.5], its proof is different in that is relies on previous comparison results for JBSDEs but not on stopping arguments. Its conditions are more restrictive than those in [Bec06], in that differentiability of g in u is assumed instead of a local Lipschitz property, but are also more general in other aspects, namely in that g depends additionally on y, z and e and we require boundedness of $f_\cdot(0,0,0)$ instead of linear growth of $\widehat{f}$ in y and $g_t(y,z,0,e) \equiv 0$. We note that [Bec06, Theorem 3.5] can however be generalized to such additional dependencies of g, as mentioned, by imposing its conditions on g uniformly over the additional arguments and following the same line of proof.*

2. *The stochastic integrals of the BSDE solution are BMO-martingales if the assumptions of Lemma 1.3 are met. In particular, this holds under the conditions of [Bec06, Theorem 3.6] where $\lambda(A) < \infty$, $\widehat{f}$ is linearly bounded in y and g is locally Lipschitz in u with $g_t(y, z, 0, e) \equiv 0$.*

Corollary 1.21. *Let $\xi \in L^\infty(\mathcal{F}_T)$ with $\xi \geq C$ for some constant $C > 0$, $K \geq 0$ and set $a(t) := C\exp(-K(T-t))$ and $b(t) = |\xi|_\infty \exp(K(T-t))$, $\forall t \in [0,T]$. Assume f satisfies (A_{fin}) for $c \leq y, y+u \leq d$ for all $c, d \in \mathbb{R}$ with $0 < c < d$, and that $|\widehat{f}_t(y,z)| \leq K|y|$ and $g_t(y,z,0,e) = 0$. Then there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f) with $Y \geq \epsilon$ for some $\epsilon > 0$. Moreover, it holds $a(t) \leq Y_t \leq b(t)$ and $\int Z\,dW$ and $U * \widetilde{\mu}$ are BMO(P)-martingales.*

Proof. This can be shown with a similar argument for the uniqueness as above: Let (Y', Z', U') be another solution to the BSDE (ξ, f) with $Y' \geq \epsilon$ for some $\epsilon > 0$. Then f satisfies (A_{fin}) for $a(t) \wedge \epsilon \leq y, y+u \leq b(t) \vee |Y'|_\infty$; hence the solutions coincide by comparison. □

Example 1.22. *As a special case of Corollary 1.21 to be applied in Section 1.4.2, setting $K := \frac{\gamma|\varphi|^2_\infty}{2(1-\gamma)^2}$ for some $\gamma \in (0,1)$ and some predictable and bounded process φ we define*

$$\begin{aligned} f_t(y,z,u) &:= \widehat{f}_t(y,z) + \int_E g_t(y,u,e)\,\zeta_t(e)\,\lambda(de) \\ &:= \frac{\gamma}{2(1-\gamma)^2}|\varphi_t|^2 y + \int_E (\frac{1}{1-\gamma}((u(e)+y)^{1-\gamma}y^\gamma - y) - u(e))\,\zeta_t(e)\,\lambda(de). \end{aligned}$$

From $\frac{\partial g}{\partial y}(t,y,u,e) = \left(\frac{u+y}{y}\right)^{1-\gamma} + \frac{\gamma}{1-\gamma}\left(\frac{u+y}{y}\right)^{-\gamma} - \frac{1}{1-\gamma}$, we see that f is Lipschitz in y within the truncation bounds. Moreover, g is continuously differentiable with bounded derivatives and

$$\frac{\partial g}{\partial u}(t,y,u,e) = \left(\frac{u+y}{y}\right)^{-\gamma} - 1 > -1,$$

for $c \leq y, y+u \leq d$.

1.3.2 The case of infinite activity

Solutions to JBSDEs with linear generators in the form

$$f_t(y,z,u) := \alpha^0_t + \alpha_t y + \beta_t z + \int_E \gamma_t(e)u(e)\,\zeta_t(e)\,\lambda(de),$$

for predictable coefficients α^0, α, β and γ admit, as usual, a representation in terms of an adjoint process Γ. In our context of bounded solutions, one needs rather weak conditions on the adjoint process. This will be used later on in Section 1.4.

Lemma 1.23. *Let f be a linear generator and $\xi \in L^\infty(\mathcal{F}_T)$.*

1. *Assume that $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ is a solution to the BSDE (ξ, f). Suppose that the adjoint process $(\Gamma^t_s)_{s\in[,t,T]} := (\exp(\int_t^s \alpha_u\, du)\mathcal{E}(\int \beta dW + \gamma * \widetilde{\mu})^s_t)_{s\in[t,T]}$ is in $\mathcal{S}^1$ for any $t \in [0, T]$ and α^0 is bounded. Then Y is represented as*

$$Y_t = E_t[\Gamma^t_T \xi + \int_t^T \Gamma^t_s \alpha^0_s\, ds]. \tag{1.15}$$

2. *Let α^0, α, β and $\widetilde{\gamma}_t := \int_E |\gamma_t(e)|^2\, \zeta_t(e)\lambda(de)$, $t \in [0, T]$, be bounded and $\gamma \geq -1$. Then there is a unique solution in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f) and (1.15) holds.*

Proof. 1. Fix $t \in [0, T]$ and denote Γ_s for Γ^t_s and $s \in [t, T]$. By Itô's formula, it follows

$$\begin{aligned}
-d(Y_s\Gamma_s) =& - Y_{s-}d\Gamma_s - \Gamma_{s-}dY_s - d[Y, \Gamma]_s \\
=& - Y_{s-}\Gamma_{s-}\alpha_s ds + \Gamma_{s-}\left(\alpha^0_s + \alpha_s Y_{s-} + \beta_s Z_s + \int_E \gamma_s(e)U_s(e)\, \zeta_s(e)\lambda(de)\right) ds \\
& - \beta_s Z_s \Gamma_{s-} ds - \Gamma_{s-}\int_E \gamma_s(e)U_s(e)\, \zeta_s(e)\lambda(de)ds - \Gamma_{s-}(Y_{s-}\beta_s + Z_s)dW_s \\
& - \Gamma_{s-}\int_E Y_{s-}\gamma_s(e) + U_s(e)(1 + \gamma_s(e))\, \widetilde{\mu}(ds, de) \\
=& \Gamma_{s-}\alpha^0_s ds - dM_s,
\end{aligned}$$

with local martingale

$$dM_s = \Gamma_{s-}(Y_{s-}\beta_s + Z_s)dW_s + \Gamma_{s-}\int_E Y_{s-}\gamma_s(e) + U_s(e)(1 + \gamma_s(e))\, \widetilde{\mu}(ds, de).$$

Using the assumptions on Y, Γ and α^0, it is easy to see that the process

$$Y_t - Y_u\Gamma^t_u - \int_t^u \Gamma^t_{s-}\alpha^0_s\, ds = M_u - M_t, \quad u \in [t, T]$$

is in $\mathcal{S}^1$. Hence M is a uniformly integrable martingale and taking conditional expectations yields (1.15).

2. It is known that there exists a unique solution $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE (ξ, f) for Lipschitz driver f and $\xi \in L^2(\mathcal{F}_T)$, and that classical a-priori estimates of the form $|Y_t| \leq cE_t[|\xi|^2 + \int_t^T |\alpha^0_s|^2\, ds]$ hold for some constant $c = c(T, \|\alpha\|_\infty, \|\beta\|_\infty, \widetilde{\gamma})$; see e.g. [Bec06, Propositions 3.2-3.3]. Therefore $Y \in \mathcal{S}^\infty$ and since the predictable process $\int_0^\cdot |\beta_s|^2\, ds + \int_0^\cdot\int_E \gamma_s(e)^2\, \nu(ds, de)$ is bounded, then the stochastic exponential $\mathcal{E}(\int \beta dW + \gamma * \widetilde{\mu})$ is in $\mathcal{S}^1$ by Theorem 2.31 in Appendix 2.5 of Chapter 2. Hence part 1. applies. □

The aim of this section is to prove existence and uniqueness beyond Theorem 1.18 for the case of possibly infinite activity of jumps, i.e. $\lambda(A)$ may be infinite, for A in (1.7). In order to show

Theorems 1.26 and 1.28, we use the monotone stability approach of [Kob00]: We approximate the generator f of the form (1.8) (with A such that $\lambda(A)=\infty$) by a sequence of generators $(f^n)_{n\in\mathbb{N}}$ of the form (1.8) (with A_n such that $\lambda(A_n)<\infty$) for which solutions exist, and we show that the limit of these solutions exist and it solves the BSDE with the original datum. By Proposition 1.24, convergence works if the terminal condition ξ is sufficient small. That is why we can not apply this Proposition directly to data $(\xi,f^n)_{n\in\mathbb{N}}$. Instead we sum (converging) solutions for sufficient small $\frac{1}{N}$-fractions of the desired terminal condition. This is inspired by the iterative ansatz from [Mor10] for a particular generator. For our general context we elaborate proofs differently, e.g. using induction arguments, but yet in a compact way. In more detail, the method in Theorem 1.26 is to construct generators $(f^{k,n})_{1\le k\le N,n\in\mathbb{N}}$ and solutions $(Y^{k,n},Z^{k,n},U^{k,n})$ to the BSDEs with data $(\xi/N,f^{k,n})$ for N sufficient large enough such that $(Y^{k,n},Z^{k,n},U^{k,n})$ converges if n tends to infinity and $(Y^n,Z^n,U^n):=\sum_{k=1}^N(Y^{k,n},Z^{k,n},U^{k,n})$ solves the BSDE (ξ,f^n). In this case (Y^n,Z^n,U^n) converges and its limit is a solution candidate for the BSDE (ξ,f). For this program, we next show a stability result for JBSDE:

Proposition 1.24. *Let $(\xi^n)\subset L^\infty(\mathcal{F}_T)$ with $\xi^n\to\xi$ in $L^2(\mathcal{F}_T)$ and $(f^n)_{n\in\mathbb{N}}$ be a sequence of generators with $f^n_\cdot(0,0,0)=0$, for all n, having property (B_{γ^n}) such that $K_f^{y,z}:=\sup_{n\in\mathbb{N}}K_{f^n}^{y,z}<\infty$. Denote by $(Y^n,Z^n,U^n)\in\mathcal{S}^\infty\times\mathcal{H}^2\times\mathcal{H}^2_\nu$ the solution to the BSDE (ξ,f^n) with Y^n bounded by $|\xi|_\infty\exp(K_{f^n}^{y,z}T)$ and set $\widetilde{c}:=|\xi|_\infty\exp(K_f^{y,z}T)$. Assume that Y^n converges pointwise, (Z^n,U^n) converges to (Z,U) weakly in $\mathcal{H}^2\times\mathcal{H}^2_\nu$ and the estimate $|f^n_t(0,0,u)|\le\widehat{K}|u|_t^2+\widehat{L}_t$ holds for all n and $u\in L^0(\mathcal{B}(E)$ with $|u|\le 2\widetilde{c}$, $\widehat{K}\in\mathbb{R}_+$ and $\widehat{L}\in L^1(P\otimes dt)$. Then (Z^n,U^n) converges to (Z,U) strongly in $\mathcal{H}^2\times\mathcal{H}^2_\nu$, if $|\xi|_\infty\equiv\widetilde{c}\exp(-K_f^{y,z}T)\le\exp(-K_f^{y,z}T)/(80\max\{K_f^{y,z},\widehat{K}\})$.*

Proof. We note that (Y^n,Z^n,U^n) is uniquely defined by Proposition 1.17. To prove strong convergence of $(Z^n)_{n\in\mathbb{N}}$ and $(U^n)_{n\in\mathbb{N}}$ we consider $\delta Y=Y^n-Y^m$, $\delta Z=Z^n-Z^m$, $\delta U=U^n-U^m$ and apply Itô's formula for general semimartingales to $(\delta Y)^2$ to obtain

$$\begin{aligned}(\delta Y_0)^2=(\delta Y_T)^2&+\int_0^T 2\delta Y_{s-}(f^n_s(Y^n_{s-},Z^n_s,U^n_s)-f^m_s(Y^m_{s-},Z^m_s,U^m_s))ds\\&-\int_0^T\|\delta Z_s\|^2ds-2\int_0^T\delta Y_{s-}\delta Z_s\,dW_s\\&-\int_0^T\!\!\int_E(\delta Y_{s-}+\delta U_s(e))^2-(\delta Y_{s-})^2\,\widetilde{\mu}(ds,de)\\&-\int_0^T\!\!\int_E(\delta Y_{s-}+\delta U_s(e))^2-(\delta Y_{s-})^2-2\delta Y_{s-}\delta U_s(e)\,\nu(ds,de).\end{aligned}$$

Noting that the stochastic integrals are martingales it follows

$$\begin{aligned}
&E\left(\int_0^T 2\delta Y_{s-}(f_s^n(Y_{s-}^n, Z_s^n, U_s^n) - f_s^m(Y_{s-}^m, Z_s^m, U_s^m))\, ds\right) \\
&\quad = E\left(\int_0^T\!\!\int_E \delta U_s(e)^2\, \nu(ds, de)\right) + E\left(\int_0^T \|\delta Z_s\|^2\, ds\right) - E((\delta Y_T)^2) + E((\delta Y_0)^2).
\end{aligned} \tag{1.16}$$

Using the inequalities $a \leq a^2 + \frac{1}{4}$, $(a+b)^2 \leq 2(a^2+b^2)$, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, the Lipschitz property of f^n in y and z and the estimate for $f_t^n(0,0,u)$, we have

$$\begin{aligned}
&|f_s^n(Y_{s-}^n, Z_s^n, U_s^n) - f_s^m(Y_{s-}^m, Z_s^m, U_s^m)| \\
&\qquad \leq K_{f^n}^{y,z}\Big(|Y_{s-}^n| + \|Z_s^n\|\Big) + K_{f^m}^{y,z}\Big(|Y_{s-}^m| + \|Z_s^m\|\Big) \\
&\qquad\quad + \widehat{K}|U_s^n|_s^2 + \widehat{L}_s + \widehat{K}|U_s^m|_s^2 + \widehat{L}_s \\
&\qquad \leq K_2\Big(\|\delta Z_s\|^2 + \|Z_s^n - Z_s\|^2 + \|Z_s\|^2 + |\delta U_s|_s^2 + |U_s^n - U_s|_s^2 + |U_s|_s^2\Big) \\
&\qquad\quad + K_1 + 2\widehat{L}_s,
\end{aligned} \tag{1.17}$$

where $K_1 := K_f^{y,z}(2\widetilde{c} + \frac{1}{2}) \in \mathbb{R}$, $K_2 := 5\max\{K_f^{y,z}, \widehat{K}\}$ and $|\cdot|_t$ is defined in (1.6). Combining inequalities (1.16) and (1.17) yields

$$\begin{aligned}
E\left(\int_0^T \|\delta Z_s\|^2 + |\delta U_s|_s^2\, ds\right) \leq & 2E\bigg(\int_0^T |\delta Y_{s-}|\Big(K_1 + 2\widehat{L}_s + K_2\big(\|\delta Z_s\|^2 + \|Z_s^n - Z_s\|^2 \\
& \qquad + \|Z_s\|^2 + |\delta U_s|_s^2 + |U_s^n - U_s|_s^2 + |U_s|_s^2\big)\Big)\, ds\bigg) \\
& + E((\xi^n - \xi^m)^2).
\end{aligned}$$

Let us recall that the predictable projection of Y, denoted by Y^p, is defined as the unique predictable process X such that $X_\tau = E_{\tau-}[Y_\tau]$ on $\{\tau < \infty\}$ for all predictable times τ. For Y^n it holds $(Y^n)^p = Y_-^n$. This follows from [JS03, Proposition I.2.35] using that Y^n is càdlàg, adapted and quasi-left-continuous, as $\Delta Y_\tau = \Delta U * \widetilde{\mu}_\tau = 0$ on $\{\tau < \infty\}$ holds for all predictable times τ thanks to the absolute continuity of the compensator ν. Noting that $1 - 2K_2|\delta Y_{s-}| \geq 1 - 4K_2\widetilde{c} \geq \frac{3}{4}$ and setting $Y := \lim_{n\to\infty} Y^n$ we deduce by the weak convergence of $(Z^n)_{n\in\mathbb{N}}$ and $(U^n)_{n\in\mathbb{N}}$, $Y_-^n = (Y^n)^p \uparrow (Y)^p$ as $n \to \infty$ and by Lebesgue's dominated convergence theorem

$$
\begin{aligned}
&\frac{3}{4}E\left(\int_0^T \|Z_s^n - Z_s\|^2 + |U_s^n - U_s|_s^2\, ds\right) \\
&\leq \frac{3}{4}\liminf_{m\to\infty} E\left(\int_0^T \|Z_s^n - Z_s^m\|^2 + |U_s^n - U_s^m|_s^2\, ds\right) \\
&\leq \liminf_{m\to\infty} 2E\left(\int_0^T |\delta Y_{s-}|(K_1 + 2\widehat{L}_s + K_2(\|Z_s^n - Z_s\|^2 + \|Z_s\|^2 + |U_s^n - U_s|_s^2 + |U_s|_s^2))\, ds\right) \\
&\qquad\quad + E((\xi^m - \xi^n)^2) \\
&= 2E\left(\int_0^T |Y_{s-}^n - (Y_s)^p|(K_1 + 2\widehat{L}_s + K_2(\|Z_s^n - Z_s\|^2 + \|Z_s\|^2 + |U_s^n - U_s|_s^2 + |U_s|_s^2))\, ds\right) \\
&\quad + E((\xi - \xi^n)^2).
\end{aligned}
$$

Taking into account that $\frac{3}{4} - 2K_2|Y_{s-}^n - (Y_s)^p| \geq \frac{3}{4} - 4K_2\widetilde{c} \geq \frac{1}{2}$ we conclude

$$
\begin{aligned}
&\frac{1}{2}\limsup_{n\to\infty} E\left(\int_0^T \|Z_s^n - Z_s\|^2 + |U_s^n - U_s|_s^2\, ds\right) \\
&\qquad \leq \limsup_{n\to\infty} 2E\left(\int_0^T |Y_{s-}^n - (Y_s)^p|(K_1 + 2\widehat{L}_s + \|Z_s\|^2 + |U_s|_s^2)\, ds\right) + E((\xi^n - \xi)^2) = 0,
\end{aligned}
$$

using again the dominated convergence theorem. □

We will need the following result which is a slight variation of Lemma 2.5 from [Kob00].

Lemma 1.25. *Let $(Z^n)_{n\in\mathbb{N}}$ be convergent in $\mathcal{H}^2$ and $(U^n)_{n\in\mathbb{N}}$ convergent in $\mathcal{H}^2_\nu$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$\sup_{n_k}\|Z^{n_k}\| \in L^2(P\otimes dt) \quad \text{and} \quad \sup_{n_k}|U_t^{n_k}|_t \in L^2(P\otimes dt).$$

Proof. The result for $(Z^n)_{n\in\mathbb{N}}$ is from [Kob00] and the argument for $(U^n)_{n\in\mathbb{N}}$ is analogous. □

Theorem 1.26 (infinite activity)**.** *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let (f^n) be a sequence of generators satisfying condition (B_{γ^n}) of Definition 1.16 with $K_f^{y,z} := \sup_{n\in\mathbb{N}} K_{f^n}^{y,z} < \infty$. Assume that*

1. *there is $(\widehat{Y}, \widehat{Z}, \widehat{U})$ in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ with $\widehat{U}$ bounded and $f_t^n(\widehat{Y}_{t-}, \widehat{Z}_t, \widehat{U}_t) \equiv 0$ for all n,*
2. *for all $u \in L^0(\mathcal{B}(E), \lambda)$ with $|u| \leq |\widehat{U}|_\infty + 2|\xi|_\infty \exp(K_f^{y,z}T)$ there exist $\widehat{K} \in (0,\infty)$ and a process $\widehat{L} \in L^1(P\otimes dt)$ such that $|f_t^n(0,0,u)| \leq \widehat{K}|u|_t^2 + \widehat{L}_t$ for each $n\in\mathbb{N}$,*
3. *the sequence $(f^n)_{n\in\mathbb{N}}$ converges pointwise and monotonically to a generator f,*

4. *there is a $BMO(P)$-martingale M such that for all truncated generator functions $f_t^{n,\widehat{c}}(y,z,u) := f_t^n((y \vee (-\widehat{c})) \wedge \widehat{c}, z, (u \vee (-2\widehat{c})) \wedge (2\widehat{c}))$ with $\widehat{c} := |\widehat{Y}|_\infty + \frac{|\widehat{U}|_\infty}{2} + \exp(K_f^{y,z}T)|\xi|_\infty$ one has the estimates $\int_t^T f_s^{n,\widehat{c}}(Y_{s-}, Z_s, U_s)\, ds \leq \langle M\rangle_T - \langle M\rangle_t$ or $-\int_t^T f_s^{n,\widehat{c}}(Y_{s-}, Z_s, U_s)\, ds \leq \langle M\rangle_T - \langle M\rangle_t$ for all $n \in \mathbb{N}$, with $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}_\nu^2$,*

5. *for all $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}_\nu^2$ and $(U^n)_{n\in\mathbb{N}} \in \mathcal{H}_\nu^2$ with $U^n \to U$ in $L^2(\widetilde{\mu})$ it holds $f^n(Y_-, Z, U^n) \longrightarrow f(Y_-, Z, U)$ in $L^1(P \otimes dt)$.*

Then

i) there exists a solution $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}_\nu^2$ for the BSDE (ξ, f), with $\int Z\, dW$ and $U * \widetilde{\mu}$ being BMO(P)-martingales, and

ii) this solution is unique if additionally f satisfies (A'_γ).

Proof. Let us first outline the overall program of the proof. We want to construct generators $(f^{k,n})_{1\leq k\leq N, n\in\mathbb{N}}$ and solutions $(Y^{k,n}, Z^{k,n}, U^{k,n})$ to the BSDEs $(\xi/N, f^{k,n})$ for N sufficient large (to employ Proposition 1.24 such that $((Y^{k,n}, Z^{k,n}, U^{k,n}))_{n\in\mathbb{N}}$ converges and $(Y^n, Z^n, U^n) := \sum_{k=1}^N (Y^{k,n}, Z^{k,n}, U^{k,n})$ solves the BSDE (ξ, f^n)). We show that if for some $k < N$ and all $1 \leq l \leq k$ and $n \in \mathbb{N}$ we have already constructed generators $(f^{l,n})_{1\leq l\leq k, n\in\mathbb{N}}$ such that there exists solutions $((Y^{l,n}, Z^{l,n}, U^{l,n}))_{n\in\mathbb{N}}$ to the BSDEs $(\xi/N, f^{l,n})$ converging for $n \to \infty$, with $|Y^{l,n}|_\infty \leq \exp(K_f^{y,z}T)|\xi|_\infty/N =: \widetilde{c}$, then for $\overline{Y}^{k,n} := \widehat{Y} + \sum_{l=1}^k Y^{l,n}$ with $\overline{Z}^{k,n}$ and $\overline{U}^{k,n}$ defined analogously and

$$f_t^{k+1,n}(y,z,u) := f_t^n(y + \overline{Y}_{t-}^{k,n}, z + \overline{Z}_t^{k,n}, u + \overline{U}_t^{k,n}) - f_t^n(\overline{Y}_{t-}^{k,n}, \overline{Z}_t^{k,n}, \overline{U}_t^{k,n}) \tag{1.18}$$

there are solutions $(Y^{k+1,n}, Z^{k+1,n}, U^{k+1,n}) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}_\nu^2$ to the BSDEs $(\xi/N, f^{k+1,n})$, converging (in n) and satisfying $|Y^{k+1,n}|_\infty \leq \widetilde{c}$. Starting with the triple $(Y^{0,n}, Z^{0,n}, U^{0,n})$ defined by $(Y^{0,n}, Z^{0,n}, U^{0,n}) := (\widehat{Y}, \widehat{Z}, \widehat{U})$, formula (1.18) gives an inductive construction of the generators $f^{k,n}$ and triples $(Y^{k,n}, Z^{k,n}, U^{k,n}) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}_\nu^2$ solving the BSDE $(\xi/N, f^{k,n})$ and converging for $n \to \infty$ with $|Y^{k,n}|_\infty \leq \widetilde{c}$ for each $n \in N$ and $1 \leq k \leq N$.
Note that $f^{k+1,n}$ is Lipschitz continuous in y and z with Lipschitz constant $K_{f^n}^{y,z}$, locally Lipschitz in u and satisfies condition $(A_{\gamma^{k+1,n}})$ with

$$\gamma_s^{k+1,n}(y,z,u,u',e) := \gamma_s^n(y + \overline{Y}_{s-}^{k,n}, z + \overline{Z}_s^{k,n}, u + \overline{U}_s^{k,n}(e), u' + \overline{U}_s^{k,n}(e), e)$$

and $f_t^{k+1,n}(0,0,0) \equiv 0$. Hence by the existence and uniqueness result for the finite activity case (see Proposition 1.17), there exists a unique solution $(Y^{k+1,n}, Z^{k+1,n}, U^{k+1,n})$ to the BSDE $(\xi/N, f^{k+1,n})$ such that $Y^{k+1,n}$ is bounded by $\widetilde{c}$.
To apply Proposition 1.24, we have to check that the sequence $(Y^{k+1,n})_{n\in\mathbb{N}}$ converges pointwise, that $(Z^{k+1,n}, U^{k+1,n})_{n\in\mathbb{N}}$ converges weakly in $\mathcal{H}^2 \times \mathcal{H}_\nu^2$ and that $f^{k+1,n}(0,0,u)$ can

be locally bounded by an affine function in $|u|^2$. Having telescoping sums in (1.18) implies that $(\overline{Y}^{l,n}, \overline{Z}^{l,n}, \overline{U}^{l,n})$ solves the BSDE $(\widehat{Y}_T + \frac{l}{N}\xi, f^n)$. By the comparison result of Proposition 1.4, the sequences $(\overline{Y}^{k,n})_{n\in\mathbb{N}}$ and $(\overline{Y}^{k+1,n})_{n\in\mathbb{N}}$ are monotonic (and bounded) in n so that finite limits $\lim_{n\to\infty} Y^{k+1,n} = \lim_{n\to\infty} \overline{Y}^{k+1,n} - \lim_{n\to\infty} \overline{Y}^{k,n}$ exists, $P \otimes dt$-a.e. By Lemma 1.3, $(\overline{Z}^{k,n}, \overline{U}^{k,n})_{n\in\mathbb{N}}$ and $(\overline{Z}^{k+1,n}, \overline{U}^{k+1,n})_{n\in\mathbb{N}}$ are bounded in $\mathcal{H}^2 \times \mathcal{H}^2_\nu$; hence $(Z^{k+1,n}, U^{k+1,n})$ is weakly convergent in $\mathcal{H}^2 \times \mathcal{H}^2_\nu$ along a subsequence which we still index by n for simplicity. Due to the Lipschitz continuity of f^n and condition 2., we get for all $|u| \le 2\widetilde{c}$ that

$$\begin{aligned}
|f_t^{k+1,n}(0,0,u)| &\le |f_t^n(\overline{Y}^{k,n}_{t-}, \overline{Z}^{k,n}_t, u + \overline{U}^{k,n}_t) - f_t^n(\overline{Y}^{k,n}_{t-}, \overline{Z}^{k,n}_t, \overline{U}^{k,n}_t)| \\
&\le 2K^{y,z}_{f^n}(|\overline{Y}^{k,n}_{t-}| + \|\overline{Z}^{k,n}_t\|) + \widehat{K}(|u + \overline{U}^{k,n}_t|^2_t + |\overline{U}^{k,n}_t|^2_t) + 2\widehat{L}_t \\
&\le 2\widehat{K}|u|^2_t + \widetilde{L}_t
\end{aligned}$$

where $\widetilde{L}_t = 2K^{y,z}_f(\widehat{c} + \sup_{n\in\mathbb{N}}\|\overline{Z}^{k,n}_t\|^2 + \frac{1}{4}) + 3\widehat{K}\sup_{n\in\mathbb{N}}|\overline{U}^{k,n}_t|^2_t + 2\widehat{L}_t$. Here we used that by induction hypothesis $(\overline{Z}^{k,n}, \overline{U}^{k,n})_n$ is convergent so that $\sup_{n\in\mathbb{N}}(\|\overline{Z}^{k,n}_t\|^2 + |\overline{U}^{k,n}_t|^2_t)$ is $P \otimes dt$-integrable by Lemma 1.25 along a subsequence which again for simplicity we still index by n. This implies that $\widetilde{L} \in L^1(P \otimes dt)$, and therefore by Proposition 1.24, the sequence $(Z^n, U^n) := (\overline{Z}^{N,n}, \overline{U}^{N,n})$ converges in $\mathcal{H}^2 \times \mathcal{H}^2_\nu$ to some (Z, U) in $\mathcal{H}^2 \times \mathcal{H}^2_\nu$ while $(Y^n) := (\overline{Y}^{N,n})$ converges to some Y. Hence, $f^n(Y^n_-, Z^n, U^n) - f^n(Y_-, Z, U^n)$ converges to 0 in $L^1(P \otimes dt)$ and by condition 5. we have $f^n(Y^n_-, Z^n, U^n) \to f(Y_-, Z, U)$ in $L^1(P \otimes dt)$. The stochastic integrals $(Z^n - Z^m) \cdot W$ and $(U^n - U^m) * \widetilde{\mu}$ belong to $\mathcal{S}^2 \subset \mathcal{S}^1$ by Doob's inequality, with $\mathcal{S}^1$-norms being bounded by a multiple of $\|Z^n - Z^m\|_{\mathcal{H}^2}$ and $\|U^n - U^m\|_{\mathcal{H}^2_\nu}$ respectively. Since $|Y^n - Y^m|_{\mathcal{S}^1}$ is dominated by

$$\|f^n(Y^n_-, Z^n, U^n) - f^m(Y^m_-, Z^m, U^m)\|_{L^1(P\otimes dt)} + C(\|Z^n - Z^m\|_{\mathcal{H}^2} + \|U^n - U^m\|_{\mathcal{H}^2_\nu})$$

for some constant $C > 0$ with the bound tending to 0 as $n, m \to 0$, we can take Y in $\mathcal{S}^1$ due to completeness of $\mathcal{S}^1$ (see [DM82], $VII.3, 64)$).
Finally, (Y, Z, U) solves the BSDE (ξ, f) since the approximating solutions $(Y^n, Z^n, U^n)_{n\in\mathbb{N}}$ of the BSDE $(\xi, f^n)_{n\in\mathbb{N}}$ converge to some $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ and $f^n(Y^n_-, Z^n, U^n)$ tends to $f(Y_-, Z, U)$ in $L^1(P \otimes dt)$. Hence, we have $\int_0^t f^n_s(Y^n_{s-}, Z^n_s, U^n_s)ds \to \int_0^t f_s(Y_{s-}, Z_s, U_s)ds$, $\int_0^t Z^n_s dW_s \to \int_0^t Z_s dW_s$ and $U^n * \widetilde{\mu}_t \to U * \widetilde{\mu}_t$ P-a.e. (along a subsequence) for all t. □

As a corollary of Theorem 1.26 we have the following result that gives conditions under which the Z-component of the JBSDE solution vanishes. This result can be proved by carrying out the whole wellposedness proof as for Theorem 1.26, but for JBSDEs solely driven by the random measure $\widetilde{\mu}$ and with generator not depending on the z-argument. Instead we have found instructive to provide a neat and straightforward argument that shows under the specific conditions that the Z-component of the JBSDE is zero. This corollary will indeed be of good use in our application Section 1.4.1.

Corollary 1.27. *Let $\mu = \mu^X$ be the random measure associated to a pure-jump process X, such that the compensated random measure $\widetilde{\mu}$ alone satisfies the weak PRP with respect to the usual filtration $\mathbb{F}^X$ of X. Let W be a d-dimensional Brownian motion independent of X and set $\mathbb{F} := \mathbb{F}^{W,X}$. Let (ξ, f) be JBSDE data satisfying the assumptions of Theorem 1.26 with $\widehat{Z} = 0$, $\xi \in L^\infty(\mathcal{F}_T^X)$, and f being $\mathcal{P}(\mathbb{F}^X) \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(L^0(\mathcal{E}))$-measurable satisfying (A'_γ). Then the JBSDE (ξ, f) admits a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$, and this solution satisfies $Z = 0$.*

Proof. Let W' be a (1-dimensional) Brownian motion independent of (W, X) and denote $\bar{W} = (W, W')$. Then $\bar{W}$ is a $(d+1)$-dimensional Brownian motion independent of X. Denote $\mathbb{F}' := \mathbb{F}^{W',X}$ and $\bar{\mathbb{F}} := \mathbb{F}^{\bar{W},X}$ the usual filtrations of (W', X) and $\bar{W}, X$. As in Example 1.1-3., $(W, \widetilde{\mu})$, $(W', \widetilde{\mu})$ and $(\bar{W}, \widetilde{\mu})$ simultaneously admit the weak PRP with respect to $\mathbb{F}, \mathbb{F}'$ and $\bar{\mathbb{F}}$. Now consider the generator function $\tilde{f}$ that does not depend on z and is defined by $\tilde{f}_t(y, u) := f_t(y, 0, u)$. Because $\widehat{Z} = 0$, the conditions of Theorem 1.26 are satisfied for $\tilde{f}^n := f^n(\cdot, 0, \cdot)$. In addition, $\tilde{f}$ satisfies condition (A'_γ) because f does. Hence since ξ is $\mathcal{F}_T^X$-measurable and $\tilde{f}$ $\mathcal{P}(\mathbb{F}^X) \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(L^0(\mathcal{E}))$-measurable, then by Theorem 1.26 the JBSDE $(\xi, \tilde{f})$ simultaneously admits unique solutions (Y, Z, U), (Y', Z', U') and $(\bar{Y}, \bar{Z}, \bar{U})$ in the respective $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$-spaces for the filtrations $\mathbb{F}, \mathbb{F}'$ and $\bar{\mathbb{F}}$. Now because $\mathbb{F}$ and $\mathbb{F}'$ are both smaller filtrations than $\bar{\mathbb{F}}$ it follows by uniqueness of $(\bar{Y}, \bar{Z}, \bar{U})$ that $Z \cdot W = Z' \cdot W' = \bar{Z} \cdot \bar{W}$, which in turn by the strong orthogonality of W and W' implies $Z = Z' = 0$. $\square$

A natural way to approximate f of the form (1.8) with $\lambda(A) = \infty$ is by taking

$$f_t^n(y, z, u) := \widehat{f}_t(y, z) + \int_{A^n} g_t(u(e), e)\, \zeta_t(e)\, \lambda(de) \tag{1.19}$$

for an increasing sequence $(A_n)_{n \in \mathbb{N}} \uparrow A$ of measurable sets with $\lambda(A_n) < \infty$ (as λ is σ-finite).

Theorem 1.28. *Let generator f be of the form (1.8) and $\xi \in L^\infty(\mathcal{F}_T)$. Let $\widehat{f}$ be Lipschitz in (y, z), let $u \mapsto g_t(u, e)$ be continuously differentiable for all (ω, t, e) with derivative $\frac{\partial g}{\partial u}$ being strictly bigger than -1 and locally bounded (in u) from above uniformly in (ω, t, e). Assume that*

1. *there exists $(\widehat{Y}, \widehat{Z}, \widehat{U}) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ such that $|\widehat{U}|_\infty < \infty$, $\widehat{f}_t(\widehat{Y}_t, \widehat{Z}_t) \equiv 0$ and $g_t(\widehat{U}_t(e), e) \equiv 0$,*
2. *function g is locally bounded in $|u|^2$ uniformly in (ω, t, e), i.e. locally (in u) there exists a K such that $|g_t(u, e)| \le K|u|^2$, and*
3. *there exists $D : \mathbb{R} \mapsto \mathbb{R}$ continuous such that $g \ge 0$ and either $\widehat{f}_t(y, z) \ge D(y)$ for $|y| \le \widehat{c} := |\widehat{Y}|_\infty + \frac{|\widehat{U}|_\infty}{2} + |\xi|_\infty \exp(K_{\widehat{f}}^{y,z} T)$, or $g \le 0$ and $\widehat{f}_t(y, z) \le D(y)$ for $|y| \le \widehat{c}$.*

Then

i) there exists a solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ and for each solution triple the stochastic integrals $\int Z\,dW$ and $U * \widetilde{\mu}$ are BMO-martingales.

ii) If moreover f satisfies the (A'_γ) condition, then the solution is unique.

Finally, the same statements hold if condition 1. is replaced by boundedness of $\widehat{f}$ and f is not depending on y, i.e. $f_t(y, z, u) = f_t(z, u)$.

Proof. We check that the assumptions of Theorem 1.26 are satisfied. Clearly conditions 1. and 2. are sufficient for assumptions 1. and 2. in Theorem 1.26. For f^n given by (1.19), the sequence (f^n) is either monotone increasing or monotone decreasing, depending on the sign of g. For the next condition 4., $f^{n,\widehat{c}}$ is bounded from above (or resp. below) by $\sup_{|y|\le\widehat{c}} D(y)$ (respectively $\inf_{|y|\le\widehat{c}} D(y)$). To show that also condition 5. of Theorem 1.26 holds, we prove that $g_t(U^n_t(e), e)\mathbb{1}_{A_n}(e)$ converge to $g_t(U_t(e), e)$ in $L^1(P \otimes \nu)$ for $U^n \to U$ in $\mathcal{H}^2_\nu$, recalling (1.1). We set $B_n := (g_t(U^n_t(e), e) - g_t(U_t(e), e))\mathbb{1}_{A_n}(e)$ and $C_n := g_t(U_t(e), e)\mathbb{1}_{A^c_n}(e)$. Both sequences $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ converge to 0 $P \otimes \nu$-a.e. since $U^n \to U$ in $L^2(P \otimes \nu)$, g is locally Lipschitz in u and $A^c_n \downarrow \emptyset$. Moreover, they are bounded by integrable random variables. In particular, B_n is bounded by $\widehat{K}(\sup_{n\in\mathbb{N}} |U^n_t(e)|^2 + |U_t(e)|^2)$ for some $\widehat{K} > 0$ which is integrable along a subsequence due to Lemma 1.25. Hence applying the dominated convergence theorem yields the desired result.
In the alternative case without the condition 1., existence is still guaranteed. Indeed, let $f_t(y, z, u) = f_t(z, u)$ and $\widehat{f}$ be bounded. Denoting $\widetilde{f}_t(z, u) := f_t(z, u) - f_t(0, 0)$ and $\widetilde{\xi} := \xi + \int_0^T f_t(0, 0)\,dt$, there exists a unique solution $(\widetilde{Y}, Z, U)$ in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ to the BSDE $(\widetilde{\xi}, \widetilde{f})$ with $\int Z\,dW$ and $U * \widetilde{\mu}$ being BMO-martingales by the first version of this theorem and noting that $g_t(0, e) \equiv 0$ and $f_t(0, 0) = \widehat{f}_t(0)$ is bounded. Taking $Y_t := \widetilde{Y}_t - \int_0^t \overline{f}_s(0, 0)\,ds$, we obtain that (Y, Z, U) solves the BSDE with the data (ξ, f). Assuming that f satisfies (A'_γ), uniqueness follows from applicability of the comparison argument in Proposition 1.4. □

Example 1.29. *function g is locally bounded in u^2 in the sense of condition 2. in Theorem 1.28 if $u \mapsto g_t(u, e)$ is twice differentiable for all (ω, t, e) with the second derivative in u being locally bounded uniformly in (ω, t, e) and $g_t(0, e) \equiv g'_t(0, e) \equiv 0$ vanishing. If moreover $g' \ge -1 + \delta$ holds, then f satisfies (A_{infi}).*

Remark 1.30. *1. Note that convexity in the z,u arguments of the generator is not required for Theorems 1.26-1.28. Sometimes in the BSDE theory, convexity of BSDE generators is assumed to show uniqueness of the associated solutions. This is the case for instance in [BH08] in the Brownian setting with unbounded terminal conditions, and also in [KTPZ15a] (see Theorem 6.3 (ii) therein) in the jump setting with bounded terminal conditions. [KTPZ15a, Theorem 6.3 (i)] also states a uniqueness result that uses* *comparison arguments similar to Proposition 1.4, but with a slightly less general (A_γ) condition (as in [Roy06]).*

2. *Results on wellposedness and comparison of JBSDEs in a setup with infinite activity of jumps are also investigated in [KTPZ15a]. However, precise meanings of the assumptions there is partly not clear to us, for instance, the definition of the Banach space structure for the set $L^p(\nu) = \bigcap_{t\in[0,T]} L^p(\nu_t)$ in their Theorem 5.4 for the case of a stochastic time-inhomogeneous compensator, or the representation Assumption 4.1 which appears to require that any local martingale is locally square integrable, which would be restrictive. In addition, many assumptions of [KTPZ15a] are stated in terms of (apparently pointwise, i.e. without exception of a null-set) inequalities on stochastic fields to hold for all (t, ω, y, z, u) from a not explicitly specified set, presumably being $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times L^2(\nu_t)$ (or $\mathcal{A}(E)$ for the u component). In any case, the combined conditions in [KTPZ15a] appear difficult to check, or might require in a way a subtle choice of versions for the postulated processes in the pointwise inequalities. It would be interesting to see examples where all their conditions can be verified. It appears their wellposedness assumptions are already somewhat restrictive for some applications with a simple model with jumps driven solely by a Poisson process; for details cf. Remark 1.34.*

1.4 Applications to optimal control problems in finance

To illustrate the scope of potential applications for the previous results on JBSDEs, we show how they can be applied to solve two distinct examples of typical optimal control problems from mathematical finance. The first example considers the portfolio optimization problem for exponential utility with an additive liability, generalizing results in [Bec06] to jumps of infinite activity. We note that result in [Mor09, Mor10] on the same problem are more general in some interesting aspects (compact constraints, stock price with jumps) whereas ours are in some other ones (multiple assets, time inhomogeneous μ, unbounded controls). While [Mor09, Mor10] built on a detailed analysis of existence for the specific (quadratic) JBSDE of this particular application, the objective of this example is to show how the general theory from Section 1.3 on a broader family of JBSDEs can be easily applied also to this utility function, which has received wide interest e.g. in indifference valuation, confer [HH09, BE09, Bec10] and many more references therein.

The second example illustrates, how a change of coordinates can transform a JBSDE from an optimal control problem, which at first appears to be beyond the technical assumptions required, into a JBSDE for which the theory of Section 1.3 can be applied to derive optimal controls from existence results for BSDEs, like in [HIM05, Sek06, Bec06]. To our best knowledge, the considered power-utility problem with jumps and multiplicative liability is solved for the first time in this spirit. Based on control theoretic arguments, [Nut12a] provides a general analysis of power utility maximization, including a characterization of the so-called opportunity process in terms of a semimartingale BSDE, whose existence is inferred from existence of the optimal value process, obtained by some other means. Such an approach typically requires convexity conditions, e.g. for convex duality methods. While convexity of generator functions is useful and appears in many applications of optimization, one may note that it is not a necessity for BSDE techniques in general, cf. e.g. [HIM05] for non-convex constraints.

Using our comparison theorem, we will provide another example in Chapter 2 by proving Theorem 2.16, where our theory applies to the good-deal valuation problem in incomplete financial markets but classical results in the literature do not. The specific result used is Proposition 1.4, which will be restated as Proposition 2.6 in that chapter to make it self-contained. Note that the Girsanov kernels of interest in Theorem 2.16 do not satisfy the bounds required for the classical (A_γ) condition as in [Mor09, Roy06, KTPZ15a], but are just so that the corresponding stochastic exponentials are martingales as needed in condition (1.9) of Proposition 1.4. For this reason, the classical comparison result for JBSDEs could not be applied directly in this case without the slight generalization provided by Proposition 1.4.

As a partly common setup for these two examples, we introduce a financial market model within the framework of Section 1.1. The market consists of one savings account, with interest rate being taken to be zero for simplicity, and k risky assets ($k \le d$) whose discounted prices evolve according to the SDE

$$dS_t = \mathrm{diag}(S_t^i)_{1\le i\le k}\sigma_t(\varphi_t dt + dW_t) =: \mathrm{diag}(S_t)dR_t, \quad t \in [0,T], \tag{1.20}$$

where the market price of risk φ is a predictable $\mathbb{R}^d$-valued and dt-integrable process, with $\varphi_t \in \mathrm{Im}\,\sigma_t^{tr} = (\mathrm{Ker}\,\sigma_t)^\perp$ for all $t \le T$, and σ is a predictable $\mathbb{R}^{k\times d}$-valued process such that σ is of full rank k (i.e. $\det(\sigma_t\sigma_t^*) \neq 0$ $P\otimes dt$-a.e.) and integrable with respect to

$$\widehat{W} := W + \int \varphi_t\,dt.$$

We take the market price of risk φ to be bounded $P\otimes dt$-a.e.. The market is free of arbitrage in the sense that the set $\mathcal{M}^e$ of equivalent local martingale measures for S is non-empty. In particular, $\mathcal{M}^e$ contains the minimal martingale measure

$$d\widehat{Q} := \mathcal{E}\left(-\int \varphi\,dW\right)_T dP = \exp\left(-\int_0^T \varphi_t\,dW_t - \frac{1}{2}\int_0^T |\varphi_t|^2\,dt\right)dP, \tag{1.21}$$

under which $\widehat{W}$ is a Brownian motion and S is a local martingale by Girsanov's theorem. Note that, even in the case $k = d$ where σ is invertible, the market (1.20) is incomplete in general since the filtration is not Brownian but carries an additional (non-trivial) random measure, cf. Example 1.1.

1.4.1 Exponential utility maximization

For the financial market model with stock price dynamics described by (1.20), we consider the utility maximization problem

$$v_t(x) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} E_t[u(X_T^{\pi,t,x} - \xi)]\,, \quad t\in[0,T], x\in\mathbb{R}, \tag{1.22}$$

with additive liability ξ for the exponential utility function $u(x) := -\exp(-\alpha x)$ with absolute risk aversion $\alpha > 0$. We would like to provide a JBSDE description of the value process v and optimal trading strategy π^* of the problem (1.22). We consider first the case without constraint on the trading strategies and with the returns on the risky asset price process being solely driven by the Brownian motion W as in (1.20), and then the case with returns driven by the compensated random measure of a pure-jump Lévy process and with constraint on the strategies.

Case with continuous risky asset price and without constraint

The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{R}^d$-valued, predictable, S-integrable processes π for which the following two conditions are satisfied:

i) $E\left(\int_0^T |\pi_t|^2\,dt\right) < \infty$,

ii) the family $\left\{\exp\left(-\alpha\int_0^\tau \pi_t d\widehat{W}_t\right)\middle|\tau \text{ stopping time}, \tau\le T\right\}$ of random variables is uniformly integrable under P.

Starting from initial capital $x\in\mathbb{R}$ at some time $t\in[0,T]$, the wealth process corresponding to investment strategy $\pi\in\mathcal{A}$ is given by $X_s^\pi = X_s^{\pi,t,x} = x + \int_t^s \pi_u\,d\widehat{W}_u$, $s\in[t,T]$.

We assume $k=d$. Let (Y,Z,U) in $\mathcal{S}^\infty(\widehat{Q})\times\mathcal{H}^2(\widehat{Q})\times\mathcal{H}^2_\nu(\widehat{Q})$ be the unique solution to the BSDE

$$Y_t = \xi + \int_t^T f_s(Y_{s-},Z_s,U_s)\,ds - \int_t^T Z_s\,d\widehat{W}_s - \int_t^T\!\!\int_E U_s(e)\,\widetilde{\mu}(ds,de)$$

under the minimal local martingale measure $\widehat{Q}$ for the generator function f given by

$$f_t(y,z,u) := -\frac{|\varphi_t|^2}{2\alpha} + \int_E \frac{\exp(\alpha u(e)) - \alpha u(e) - 1}{\alpha}\,\zeta_t(e)\,\lambda(de)\,, \tag{1.23}$$

which does exist by Theorem 1.28. Under P the BSDE is of the form

$$Y_t = \xi + \int_t^T f_s(Y_{s-},Z_s,U_s) - \varphi_s Z_s\,ds - \int_t^T Z_s\,dW_s - \int_t^T\!\!\int_E U_s(e)\,\widetilde{\mu}(ds,de).$$

To show optimality via the classical martingale principle for optimality one constructs, cf. [HIM05, Sek06], a family of processes $(V^\pi)_{\pi\in\mathcal{A}}$ such that

(i) $V_t^\pi = V_t$ is a fixed $\mathcal{F}_t$-measurable bounded random variable, invariant over $\pi \in \mathcal{A}$,

(ii) $V_T^\pi = -\exp(-\alpha(X_T^\pi - \xi)) = -\exp(-\alpha(x + \int_t^T \pi_s d\widehat{W}_s - \xi))$,

(iii) V^π is a supermartingale for all $\pi \in \mathcal{A}$ and there exists a $\pi^* \in \mathcal{A}$ such that V^{π^*} is a martingale

under P over the time interval $[t,T]$. Then π^* is the optimal strategy and $(V_s^{\pi^*})_{s\in[t,T]}$ is the value process of the control problem (1.22). Indeed, $E_t[V_T^\pi] \le V_t^\pi = V_t^{\pi^*} = E_t[V_T^{\pi^*}]$ for each $\pi \in \mathcal{A}$ implies $v_t(x) = \text{ess sup}_{\pi\in\mathcal{A}} E_t[V_T^\pi] = V_t^{\pi^*}$. An ansatz $V^\pi = u(X^\pi - Y)$ yields

$$V_s^\pi = V_t^\pi \exp\left(\frac{\alpha^2}{2}\int_t^s \left|\pi_r - Z_r - \frac{\varphi_r}{\alpha}\right|^2 dr\right)\mathcal{E}(M)_t^s \quad \text{for all } s\in[t,T], \quad \text{with}$$

$$M_t = -\alpha\int_0^t \pi_r - Z_r\, d\widehat{W}_r + \int_0^t\int_E \exp(\alpha U_r(e) - 1)\,\widetilde{\mu}(dr,de) \quad \text{and} \quad \mathcal{E}(M)_t^s := \frac{\mathcal{E}(M)_s}{\mathcal{E}(M)_t}.$$

Therefore, V^π is a supermartingale for all $\pi \in \mathcal{A}$ and a martingale for $\pi^* = Z + \varphi/\alpha$ due to the fact that $\mathcal{E}(M)$ is a (local) martingale of the form

$$\mathcal{E}(M)_s = \exp\left(-\frac{\alpha^2}{2}\int_0^s |\pi_u - Z_u - \varphi_u/\alpha|^2\, du\right)\exp\left(-\alpha(Y_0 + \int_0^s \pi_u\, d\widehat{W}_u - Y_s)\right).$$

Using the boundedness of Y and the definition of $\mathcal{A}$, one easily concludes that $\mathcal{E}(M)$ is uniformly integrable and hence a martingale (cf. [Bec06, Equation (4.19)]). This yields

Example 1.31. *For $k = d$, let $(Y,Z,U) \in \mathcal{S}^\infty(\widehat{Q}) \times \mathcal{H}^2(\widehat{Q}) \times \mathcal{H}^2_\nu(\widehat{Q})$ be the unique solution to the BSDE (ξ, f) under $\widehat{Q}$ for generator f from (1.23). Then the strategy $\pi^* = Z + \varphi/\alpha$ is optimal for the control problem (1.22) and achieves at any time $t \in [0,T]$ the maximal expected exponential utility $v_t(x) = -\exp(-\alpha(x - Y_t)) = V_t^{\pi^*}$.*

Case with discontinuous risky asset price and constraints

For illustration and also to show the extend to which our main results can be used to recover the ones of [Mor09, Mor10] in a setting with pure-jump stock price, we consider again the problem (1.22) of exponential utility maximization, but now in a financial market with constraints on the trading strategies and with pure-jump stock price (as e.g. in the CGMY model of [CGMY02]) possibly of infinite jump activity. In particular for some non-convex constraints on strategies, we obtain the solution in terms of a JBSDE with non-convex generator in u; cf. Remark 1.33. This shows how our results can be applied to situations where the JBSDE generators are not convex and the wellposedness results e.g. of [LS14] may not be applicable. Finally we provide some details to part of Remark 1.30.2., showing in a simple setup how the assumptions of [KTPZ15a] may not cover the JBSDE for the entropic risk measure which is also an important financial application with nice connections to exponential utility maximization.

For our setup, let $\mu = \mu^L$ be the random measure associated to a pure-jump Lévy process L possibly of infinite activity (e.g. a Gamma process) with Lévy measure λ and independent of the Brownian motion W. Denote $\mathbb{F} = \mathbb{F}^L$ the usual filtration of L and $\mathbb{F} := \mathbb{F}^{W,L}$ the usual filtration generated by L and W. Is is known that W and the compensated random measure $\widetilde{\mu} = \widetilde{\mu}^L := \mu^L - \nu$ have the weak PRP with respect to $\mathbb{F}$ (see Example 1.1-1.,3.). Contrary to the setup of Section 1.4.1 where the risky asset returns are driven solely by W, we consider here a financial market with a single stock with pure-jump price process S modelled as

$$dS_t = S_{t-}(\beta_t dt + \int_E \psi_t(e))\widetilde{\mu}(dt, de), \quad t \in [0, T],$$

where $E = \mathbb{R} \setminus \{0\}$, β is $\mathbb{F}^L$-predictable and bounded, and $\psi > -1$ is $\widetilde{\mathcal{P}}(\mathbb{F}^L)$-measurable with ψ in $L^2(P \otimes \lambda \otimes dt) \cap L^\infty(P \otimes \lambda \otimes dt)$ and satisfying $\int_E |\psi_t(e)|^2 \lambda(de) <$ const. $P \otimes dt$-a.e.. The set $\mathcal{A}$ of admissible trading strategies consists of all $\mathbb{R}$-valued predictable S-integrable processes $\pi \in L^2(P \otimes dt)$, such that $\pi_t(\omega) \in C$ for all (t, ω), for a fixed compact constraint set $C \subset \mathbb{R}$. For a strategy $\pi \in \mathcal{A}$, the corresponding wealth process for initial capital x at time t is

$$X_s^{\pi,t,x} = X_t^{\pi,t,x} + \int_t^s \pi_u \frac{dS_u}{S_{u-}} = x + \int_t^s \pi_u \Big(\beta_u du + \int_E \psi_u(e)\widetilde{\mu}(du, de)\Big), \quad s \geq t.$$

We consider again the exponential utility maximization problem (1.22) but with liability $\xi \in L^\infty(\mathcal{F}_T^L)$. Because of the compactness of C and the integrability conditions on ψ, admissible strategies are bounded and for all $\pi \in \mathcal{A}$ holds

$$\{\exp(-\alpha X_\tau^\pi),\ \tau\ \mathbb{F}\text{-stopping time}\} \text{ is uniformly integrable,}$$

the details being analogous to Lemma 1 in [Mor10]. Consider the JBSDE

$$-dY_t = f(t, U_t)dt - \int_E U_t(e)\widetilde{\mu}(dt, de), \quad Y_T = \xi, \tag{1.24}$$

with generator f given by

$$f(t, u) := \inf_{\pi \in C} \Big(-\pi\beta_t + \int_E g_\alpha(u(e) - \pi\psi_t(e))\lambda(de)\Big), \quad t \in [0, T], \tag{1.25}$$

for the function $g_\alpha : \mathbb{R} \to \mathbb{R}$ defined by $g_\alpha(u) := \frac{e^{\alpha u} - \alpha u - 1}{\alpha}$. By assumptions on β and ψ, the generator function f is $\mathbb{F}^L$-predictable in (t, ω). Note that the JBSDE for the entropic risk measure of Remark 1.30 is a particular case of the above for the stock price being constant, i.e. for $\beta \equiv \psi \equiv 0$.

Example 1.32. *Let $(Y,U) \in \mathcal{S}^\infty \times \mathcal{H}^2_\nu$ be the unique solution to the JBSDE (1.24). Then the strategy π^* such that π^*_t achieves the infimum in (1.25) for $f(t,U_t)$ is optimal for the control problem (1.22) and achieves at any $t \in [0,T]$ the maximal expected exponential utility $v_t(x) = -\exp(-\alpha(x - Y_t)) = V_t^{\pi^*}$.*

Proof for Example 1.32: Using the martingale optimality principle, one shows analogously to Section 1.4.1 that if $(Y,U) \in \mathcal{S}^\infty \times \mathcal{H}^2_\nu$ is solution to the JBSDE (1.24) then the solution to the utility maximization problem (1.22) is indeed given by $v_t(x) = u(x - Y_t)$, with optimal strategy the process π^* such that $\pi^*_t(\omega)$ achieves the infimum in (1.25) for the generator function f computed at $(t, U_t(\omega))$ for all (t,ω). It thus remains to show that the JBSDE (1.24) indeed admits a unique solution. This will be shown by applying Theorem 1.26 and its Corollary 1.27 since $\xi \in L^\infty(\mathcal{F}^L_T)$ and f is $\mathbb{F}^L$-predictable in (t,ω). For this purpose, consider a sequence $(f^n)_n$ of generators functions with

$$f^n(t,u) := \inf_{\pi \in C} \Big(-\pi\beta_t + \int_{A_n} g_\alpha(u(e) - \pi\psi_t(e))\lambda(de)\Big),$$

where $(A_n)_n$ is a sequence of measurable sets with $A_n \uparrow E$ and $\lambda(A_n) < \infty$ for all $n \in \mathbb{N}$; typically $A_n = (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, +\infty)$. We first show that f^n satisfies a condition (B^{γ^n}). Each f^n is independent of (y,z), hence $K^{y,z}_f = K^{y,z}_{f^n} = 0$ for each n. Since g_α is continuously differentiable with locally bounded derivative and $\lambda(A_n) < \infty$, then the function

$$L^2(\lambda) \ni u \mapsto \int_{A_n} g_\alpha(u(e) - \pi\psi_t(e))\lambda(de)$$

is locally Lipschitz in $u \in L^2(\lambda)$, with a Lipschitz constant $K^{n,c}$ on $\{u : |u| \le c\}$, for $c > 0$, independent of $\pi \in C$. Hence f^n is locally Lipschitz in $u \in L^2(\lambda)$ for all n, and by uniform boundedness of C, β, ψ, it holds that $f^n(\cdot, 0)$ is bounded for all n. It remains to show that for each n there exists γ^n such that f^n satisfies (A_{γ^n}). For $\pi \in C$, define the function $f^{\pi,n}$ by

$$f^{\pi,n}(t,u) := -\pi\beta_t + \int_{A_n} g_\alpha(u(e) - \pi\psi_t(e))\lambda(de).$$

By Example 1.8 and Example 1.10-2., the functions $f^{\pi,n}$ satisfy $(A_{\gamma^{\pi,n}})$ for

$$\gamma_t^{\pi,n,u,u'}(e) := \mathbf{1}_{A_n} \int_0^1 g'_\alpha\Big(l(u(e) - \pi\psi_t(e)) + (1-l)(u'(e) - \pi\psi_t(e))\Big)dl.$$

It follows that $f^n = \inf_{\pi\in C} f^{\pi,n}$ satisfies (A_{γ^n}) for

$$\gamma_t^{n,u,u'}(e) := \sup_{\pi\in C} \gamma_t^{\pi,n,u,u'}(e)\mathbf{1}_{\{u \ge u'\}}(e) + \inf_{\pi \in C} \gamma_t^{\pi,n,u,u'}(e)\mathbf{1}_{\{u < u'\}}(e).$$

Now since $g'_\alpha > -1$ and g'_α is locally bounded from above in u, then for U, U' with $|U|_\infty < \infty$, $|U'|_\infty < \infty$ that there exists $\delta \in (0,1)$ and a constant $c > 0$ such that $\bar{\gamma}^n_t(e) :=$

$\bar{\gamma}_t^{n,U,U'}(e) \in [-1+\delta, c]$, $P \otimes \lambda \otimes dt$-a.e.. Finally because $\lambda(A_n) < \infty$, f^n satisfies condition (A_{γ^n}) and thus (B_{γ^n}) for all n.

We next check conditions 1.-5. of Theorem 1.26 for f and that f satisfies in addition (A'_γ):
Condition 1.: Since f^n is independent of (y,z), condition 1. can be replaced by considering instead the generators $\tilde{f}^n := f^n - f^n(\cdot, 0)$ which clearly satisfy $\tilde{f}^n(t,0) \equiv 0,\ t \in [0,T]$ and proceeding analogously to the proof of the last statement of Theorem 1.28. Note for this purpose that $f^n(\cdot,0)$ and $f(\cdot,0)$ are bounded (since β, ψ, π are) and the function g_α is locally of quadratic growth (cf. in condition 2. below).

Condition 2.: By Taylor's inequality the function g_α is of quadratic growth locally in u, i.e. for any $c > 0$ there exists $K(c) \in (0,\infty)$ such that $g_\alpha(u) \le K(c)|u|^2$ for all $u \in [-c,c]$. This implies that for all $c > 0$, there exists $\tilde{K}(c) > 0$ independent of $\pi \in C$ and $n \in \mathbb{N}$ such that for all $u \in L^2(\lambda)$ with $|u| \le c$ λ-a.e. one has for any n

$$|f^{\pi,n}(t,u)| \le \text{const.} + \tilde{K}(c)|u|^2_{L^2(\lambda)}, \quad t \in [0,T]. \tag{1.26}$$

Now using the inequality $|\inf_\pi f^{\pi,n}| \le \sup_\pi |f^{\pi,n}|$ implies condition 2.

Condition 3.: First note that $|\inf_\pi f^{\pi,n} - \inf_\pi f^\pi| \le \sup_\pi |f^{\pi,n} - f^\pi|$, with f^π being the analogue of $f^{\pi,n}$ for f. Using this inequality leads for all $t \in [0,T]$ to

$$|f_t^n(u) - f_t(u)| \le \sup_{\pi \in C} \int_E g_\alpha(u(e) - \pi\psi_t(e)) \mathbf{1}_{(A_n)^c}(e)\lambda(de).$$

Since $A_n \uparrow E$, then $\sup_{\pi \in C} g_\alpha(u - \pi\psi_t)\mathbf{1}_{(A_n)^c} \downarrow 0$. Moreover since g_α is locally quadratic in u, then for $u \in L^2(\lambda) \cap L^\infty(\lambda)$ one has

$$\sup_{\pi \in C} g_\alpha(u - \pi\psi_t)\mathbf{1}_{(A_n)^c} \le \text{const.}\left(|u|^2 + |\psi_t|^2\right). \tag{1.27}$$

By assumption $\psi_t \in L^2(\lambda)$, and hence dominated convergence implies from (1.27) that $(f^n)_n$ converges pointwise to f. Indeed $(f^n)_n$ increases to f since $A_n \uparrow E$ and $g_\alpha \ge 0$.

Condition 4.: Since $g_\alpha \ge 0$, one has $f^{n,\pi}(t,u) \ge -\pi\beta_t$ for any u, π and t, which implies $f^n(t,u) \ge -|\beta|_\infty \text{diam}(C)$ for any $u \in L^2(\lambda)$ and $\pi \in \mathcal{A}$, where $\mathrm{diam}(C) := \sup_{x,y \in C}|x-y|$ denotes the diameter of the set C. Condition 4. now holds for any BMO-martingale M with $\langle M \rangle_t = |\beta|_\infty \text{diam}(C) \cdot t$.

Condition 5.: Let $U^n, U \in \mathcal{H}^2_\nu$ with $|U^n|_\infty < \infty, |U|_\infty < \infty$ such that U^n converges to U in $\mathcal{H}^2_\nu$. Then as in the verification of condition 3., we have

$$\begin{aligned}
|f_t^n(U_t^n) - f_t(U_t)| &\le \sup_{\pi \in C} \left| \int_{A_n} g_\alpha(U_t^n(e) - \pi\psi_t(e))\lambda(de) - \int_E g_\alpha(U_t(e) - \pi\psi_t(e))\lambda(de) \right| \\
&\le \sup_{\pi \in C} \int_{A_n} \Big| g_\alpha(U_t^n(e) - \pi\psi_t(e)) - g_\alpha(U_t(e) - \pi\psi_t(e)) \Big| \lambda(de) \\
&\quad + \sup_{\pi \in C} \int_{(A_n)^c} g_\alpha(U_t(e) - \pi\psi_t(e))\lambda(de).
\end{aligned}$$

Consider $(b^n)_n$ and $(c_n)_n$ defined by

$$b_t^n(e) := \sup_{\pi\in\mathcal{A}} \left|g_\alpha(U_t^n(e) - \pi\psi_t(e)) - g_\alpha(U_t(e) - \pi\psi_t(e))\right|\mathbf{1}_{A_n}$$

and $c_t^n(e) := \sup_{\pi\in\mathcal{A}} g_\alpha(U_t(e) - \pi\psi_t(e))\mathbf{1}_{(A_n)^c}$. Since g_α is locally Lipschitz, $U^n \to U$ in $\mathcal{H}^2_\nu$, $A_n \uparrow E$ and C, ψ are bounded, then b^n and c^n converge to 0, $P \otimes \lambda \otimes dt$-a.e. up to a subsequence. By [AB06, Theorem 13.6] there exists a renamed subsequence $(U^n)_n$ and a function $\tilde{U} \in \mathcal{H}^2_\nu$ such that $|U^n| \le \tilde{U}$ $P \otimes \lambda \times dt$-a.e.. From local quadradicity of g_α holds a.e.

$$b_t^n(e) \le \text{const.}\big(|\tilde{U}_t(e)|^2 + |U_t(e)|^2 + |\psi_t(e)|^2\big) \quad \text{and} \quad c_t^n(e) \le \text{const.}\big(|U_t(e)|^2 + |\psi_t(e)|^2\big).$$

Condition 5. now follows by dominated convergence.

Finally to show that f satisfies condition (A'_γ), consider $\gamma^{u,u'}$ given by

$$\gamma_t^{u,u'}(e) := \sup_{\pi\in C} \gamma_t^{\pi,u,u'}(e)\mathbf{1}_{\{u\ge u'\}}(e) + \inf_{\pi\in C} \gamma_t^{\pi,u,u'}(e)\mathbf{1}_{\{u<u'\}}(e),$$

for $\gamma_t^{\pi,u,u'}(e) := \int_0^1 g'_\alpha\big(l(u(e) - \pi\psi_t(e)) + (1-l)(u'(e) - \pi\psi_t(e))\big)dl$. Then as before, by the mean-value theorem one has for all U, U' with $|U|_\infty < \infty, |U'|_\infty < \infty$ that

$$f(t, U_t) - f(t, U'_t) \le \int_E \gamma_t^{U,U'}(e)\big(U_t(e) - U'_t(e)\big)\lambda(de).$$

Let u, u' be bounded by $c > 0$. Then since $g'_\alpha(0) = 0$, the mean-value theorem applied to g'_α yields $|\gamma_t^{\pi,u,u'}(e)| \le \sup_{|u|\le\tilde{c}}|g''_\alpha(s,u)|\Big(|u - \pi\psi_t(e)| + |u' - \pi\psi_t(e)|\Big)$, for all $\pi \in C$, where $\tilde{c} := c + \|\psi\|_\infty \text{diam}(C)$. Hence

$$\sup_{\pi\in C}|\gamma_t^{\pi,u,u'}(e)| \le \sup_{|u|\le\tilde{c}}|g''_\alpha(s,u)|\Big(|U_t(e)| + |U'_t(e)| + 2\text{diam}(C)|\psi_t(e)|\Big).$$

Now from $|\inf_\pi \gamma^\pi| \le \sup_\pi|\gamma^\pi|$, $|\sup_\pi \gamma^\pi| \le \sup_\pi|\gamma^\pi|$ and the fact that $\psi * \widetilde{\mu}$ is a BMO-martingale (since ψ is bounded and $\int_E|\psi_t(e)|^2\lambda(de) <$ const., $P \otimes dt$-a.s. by assumption), one obtains that $\gamma^{\pi,U,U'} * \widetilde{\mu}$ is a BMO-martingale if $U * \widetilde{\mu}$ and $U' * \widetilde{\mu}$ are, with $|U|_\infty < \infty$ and $|U'|_\infty < \infty$. This concludes (A'_γ) for f.

Conditions 1.-5. and (A'_γ) imply by Theorem 1.26 and Corollary 1.27 that there exists a unique solution $(Y, U) \in \mathcal{S}^\infty \times \mathcal{H}^2_\nu$ to the JBSDE (1.24), such that $U * \widetilde{\mu}$ is a BMO-martingale. $\square$

Remark 1.33. *Note that since the function $(u, \pi) \mapsto f^\pi(\cdot, u)$ is convex, then the generator $f = \inf_{\pi\in C} f^\pi(\cdot, u)$ would be convex in u if C were a convex set. There are some results in the literature ([LS14, Theorem A.28], [KTPZ15a, Theorem 6.3 (ii)]) of JBSDEs possibly of infinite activity, that require convexity of the generator function of the JBSDE in order to guarantee wellposedness of such. Our theoretical results do not require this feature of the generator. We now provide a concrete counter example for which f in (1.25) is not convex* *in u. A necessary condition for this is for the constraint set not to be convex. In particular we consider for instance that C is a finite set given as $C = \{\pi^1, \ldots, \pi^m\} \subset \mathbb{R}$, for π^k being constant values that strategies could attain, including zero (i.e. $0 \in C$). Here the generator f of JBSDE (1.24) is*

$$f(t,u) = \inf_{k\in\{1,\ldots,m\}} \Big(-\pi^k\beta_t + \int_E g_\alpha(u(e) - \pi^k\psi_t(e))\lambda(de)\Big).$$

Let us consider the very simple case $\lambda(de) = \delta_{\{1\}}(de)$, for the Dirac measure concentrated at 1, i.e. L is a simple Poisson process hence with jumps of finite activity. Assuming that $\alpha = 1$ and $\beta = 0$, we thus obtain $f(t,u) = \min_{k\in\{1,\ldots,m\}} \Big(e^{(u-\pi^k\psi_t)} - (u - \pi^k\psi_t) - 1\Big)$, which is readily seen to be non-convex in $u \in \mathbb{R}$ unless $\psi \equiv 0$ or $C = \{0\}$, but satisfies the assumptions of our Theorem 1.26.

Remark 1.34. *For L being a simple Poisson process as in Remark 1.33, i.e. with $\lambda(de) = \delta_{\{1\}}(de)$, and for $\beta = 0$ and $\psi = 0$ (i.e. constant stock price), the generator function f in (1.25) coincides with the function g_α, identifying $\mathbb{R}$ with $L^2(\lambda)$. In this case the Y-component of the solution to the JBSDE (1.24) is, formally speaking, the entropic risk measure given by $Y_t = \frac{1}{\alpha}\ln\left(E[e^{\alpha\xi}\,|\mathcal{F}_t^L]\right)$, $t \in [0,T]$. The key assumptions for wellposedness in [KTPZ15a] being their Assumptions 4.3.(iii) and 5.1.(iii) (cf. Theorem 4.7 and Theorem 5.4 therein), it turns out that none them holds for the generator of the JBSDE (1.24) in the current simplified setup. Note that Assumption 4.3.(iii) requires existence of a BMO-integrand θ (denoted ψ in their paper) for $\tilde{\mu}(dt) = L - dt$ such that for $u' = 0$ and for all $u \in \mathbb{R}$ one has $|f(u) - \theta_t u| \le C|u|^2$ for some constant $C > 0$. If this were true, it would imply that the function $\mathbb{R}\setminus\{0\} \ni u \mapsto (g_\alpha(u) - \theta_t u)/|u|^2$ is bounded. But the latter cannot hold since*

$$\lim_{u\to+\infty} \frac{g_\alpha(u) - \theta_t u}{|u|^2} = \lim_{u\to+\infty} \frac{e^{\alpha u}}{|u|^2} = +\infty.$$

Hence Assumption 4.3.(iii) of [KTPZ15a] is violated for the generator function f. In addition, Assumption 5.1.(iii) requiring the functions $u \mapsto g'_\alpha(u) = (e^{\alpha u} - 1)$ to be at most of linear growth and $u \mapsto g''_\alpha(u) = \alpha e^{\alpha u}$ to be bounded on the whole real line can also not hold since both functions are actually of exponential growth. Thus, these assumptions of [KTPZ15a] seem to prevent direct application to the entropic risk measure in general; the latter is clearly covered by the stability approach as in [Mor09, Mor10] (for the specific generator in question) or as from our analysis (for the examples presented above which do not require a quadratic z-part in the generator). Overall, one can conclude that although our results do not imply those of [KTPZ15a], since we consider Lipschitz generators in z and they quadratic ones, theirs do not imply ours either.

1.4.2 Power utility maximization

Again for the market with stock price dynamics (1.20), we consider the utility optimization problem

$$v_t(x) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} E_t[u(X_T^{\pi,t,x})\xi] = E_t[u(X_T^{\pi,t,x}\xi')]\,, \quad t\in[0,T],\ x>0, \tag{1.28}$$

for power utility $u(x)=\frac{x^\gamma}{\gamma}$ with relative risk aversion $1-\gamma>0$ for $\gamma\in(0,1)$, with multiplicative liability ξ (for instance, $\xi'=(\gamma\xi)^{1/\gamma}$ may reflect an unknown future tax rate). We parametrize strategies π by fractions of wealth invested. Then the respective wealth process $X^\pi = X^{\pi,t,x}$ is $X_s^{\pi,t,x} = x+\int_t^s X_u^\pi \pi_u\,d\widehat{W}_u = x\mathcal{E}\big(\int \pi d\widehat{W}\big)_t^s$ for $s\in[t,T]$ and for π from a set $\mathcal{A}$. The set of strategies $\mathcal{A}$ is given by all $\mathbb{R}^d$-valued, predictable, S-integrable processes such that $\int \pi\,dW$ is a $BMO(P)$-martingale, cf. [HWY92].

Proposition 1.35. *Let $k=d$. Assume that there is a solution $(Y,Z,U)\in\mathcal{S}^\infty\times\mathcal{H}^2\times\mathcal{H}^2_\nu$ to the BSDE (ξ,f) with $f_t(y,z,u):=\frac{y}{2}\frac{\gamma}{1-\gamma}|\varphi_t+y/z|^2$ and $\int Z\,dW\in BMO(P)$ and where ξ is in $L^\infty(\mathcal{F}_T)$ with $\xi\ge c$ for some $c>0$. Then $Y\ge c$ holds and it holds for $V_s^\pi := U(X_s^\pi)Y_s$ that V^π is a supermartingale for all π in $\mathcal{A}$ and V^{π^*} is a martingale for $\pi^*:=\frac{1}{1-\gamma}(\varphi+Z/Y_-)\in\mathcal{A}$.*

Proof. Clearly, V^π is adapted. By a criterion from Kazamaki $\mathcal{E}\big(\int_0^\cdot \gamma\pi_u d\widehat{W}_u\big)$ is an r-integrable martingale for some $r>1$. Hence $\sup_{t\le s\le T}\mathcal{E}\big(\int_0^\cdot \gamma\pi_u d\widehat{W}_u\big)_t^s$ is integrable by Doob's inequality. Using the estimate

$$\mathcal{E}\Big(\int_0^\cdot \pi_u\,d\widehat{W}_u\Big)^\gamma = \mathcal{E}\Big(\int_0^\cdot \gamma\pi_u\,d\widehat{W}_u\Big)\exp\Big(-\frac12\gamma(1-\gamma)\int_0^\cdot |\pi_u|^2\,du\Big) \le \mathcal{E}\Big(\int_0^\cdot \gamma\pi_u\,d\widehat{W}_u\Big)\,,$$

we conclude that V^π is dominated by $\sup_{t\le s\le T}U(X_s^\pi)|Y|_\infty\in L^1(P)$. By Itô's formula, dV_s^π equals a local martingale plus the finite variation part

$$U(X_s^\pi)\Big(-f_s(Y_{s-},Z_s,U_s)+\gamma\Big(Y_{s-}\Big(\pi_s\varphi_s+\frac12(\gamma-1)|\pi_s|^2\Big)+\pi_s Z_s\Big)\Big)\,ds\,.$$

The latter part is decreasing for all $\pi\in\mathcal{A}$ and vanishes at zero for $\pi=\pi^*$. Hence V^π is a local (super-)martingale. Uniform integrability of V^π yields the (super-)martingale property. By the classical martingale optimality principle of optimal control follows

$$v_t(x) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} E_t[U(X_T^\pi\xi^{\frac1\gamma})] = V_t^{\pi^*} = \frac{x^\gamma}{\gamma}Y_t\,,$$

and evaluating at $\pi\equiv 0$ yields $\frac{x^\gamma}{\gamma}E_t[\xi]\le\frac{x^\gamma}{\gamma}Y_t$ and hence $Y\ge c$. Note that π^* is in $\mathcal{A}$ since φ is bounded, Y is bounded away from 0 and Z is an BMO integrand. $\square$

Let (Y, Z, U) be a solution to the BSDE (ξ, f) with the above data. Since a suitable solution theory for quadratic BSDEs with jumps is not available, we transform coordinates by letting

$$\widetilde{Y}_t := Y_t^{\frac{1}{1-\gamma}}, \widetilde{Z}_t := \frac{1}{1-\gamma} Y_{t-}^{\frac{\gamma}{1-\gamma}} Z_t \quad \text{and} \quad \widetilde{U}_t := (Y_{t-} + U_t)^{\frac{1}{1-\gamma}} - Y_{t-}^{\frac{1}{1-\gamma}}, \tag{1.29}$$

such that $(\widetilde{Y}, \widetilde{Z}, \widetilde{U})$ solves the BSDE for data $(\widetilde{\xi}, \widetilde{f})$ with $\widetilde{\xi} = \xi^{\frac{1}{1-\gamma}}$ and generator $\widetilde{f}_t(y, z, u)$ given by

$$\frac{\gamma|\varphi_t|^2}{2(1-\gamma)^2} y + \frac{\gamma}{1-\gamma}\varphi_t z + \int_E \left(\frac{1}{1-\gamma} \Big((u(e) + y)^{1-\gamma} y^{\gamma} - y \Big) - u(e) \right) \zeta_t(e)\, \lambda(de).$$

Looking at the proof of Lemma 1.2, we may assume that $U + Y_-$ coincides pointwise with the process Y_- or Y so that the above transformation is well-defined due to $Y \geq c$. In fact, (1.29) gives a bijection between solutions with positive Y-components to the BSDEs (ξ, f) and $(\widetilde{\xi}, \widetilde{f})$ in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$.

Next, we show the existence of a JBSDE solution for data (ξ, f) with $\xi \geq c$ for some $c > 0$. Under the probability measure $d\widetilde{P} := \mathcal{E}\Big(\frac{\gamma}{1-\gamma}\int_0^\cdot \varphi_t\, dW_t\Big)_T dP$ the process $\widetilde{W} = W - \int_0^\cdot \frac{\gamma\varphi_t}{1-\gamma}\, dt$ is a Brownian motion and the JBSDE

$$\widetilde{Y}_t = \widetilde{\xi} + \int_t^T \widetilde{f}_s(\widetilde{Y}_{s-}, \widetilde{Z}_s, \widetilde{U}_s)\, ds - \int_t^T \widetilde{Z}_s\, dW_s - \int_t^T \int_E \widetilde{U}_s(e)\, \widetilde{\mu}(ds, de)$$

under P is of the following form under $\widetilde{P}$

$$\widetilde{Y}_t = \widetilde{\xi} + \int_t^T \Big(\widetilde{f}_s(\widetilde{Y}_{s-}, \widetilde{Z}_s, \widetilde{U}_s) - \frac{\gamma\varphi_s}{1-\gamma}\widetilde{Z}_s\Big) ds - \int_t^T \widetilde{Z}_s\, d\widetilde{W}_s - \int_t^T \int_E \widetilde{U}_s(e)\, \widetilde{\mu}(ds, de), \tag{1.30}$$

noting that ν is the compensator of μ under P and $\widetilde{P}$ as well. In fact, we have

Lemma 1.36. *Assume $\lambda(E) < \infty$. Then $(\widetilde{Y}, \widetilde{Z}, \widetilde{U}) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ solves the BSDE $(\widetilde{\xi}, \widetilde{f})$ such that $\int \widetilde{Z}\, dW$ is in $BMO(P)$ if and only if $(\widetilde{Y}, \widetilde{Z}, \widetilde{U}) \in \mathcal{S}^\infty(\widetilde{P}) \times \mathcal{H}^2(\widetilde{P}) \times \mathcal{H}^2_\nu(\widetilde{P})$ solves the BSDE $\Big(\widetilde{\xi}, \widetilde{f}(y, z, u) - \frac{\varphi\gamma}{1-\gamma} z\Big)$ such that $\int \widetilde{Z}\, d\widetilde{W}$ is in $BMO(\widetilde{P})$.*

Proof. Equivalence of P and $\widetilde{P}$ imply that $\widetilde{Y} \in \mathcal{S}^\infty$ if and only if (iff) $\widetilde{Y} \in \mathcal{S}^\infty(\widetilde{P})$. Assuming that $\lambda(E) < \infty$, $\widetilde{U} \in \mathcal{H}^2_\nu$ iff $\widetilde{U} \in \mathcal{H}^2_\nu(\widetilde{P})$ due to the boundedness of $\widetilde{U}$. By [Kaz94, Theorem 3.6], the restriction of the Girsanov transform

$$\Phi : \mathcal{M}_c^{loc,0}(P) \longrightarrow \mathcal{M}_c^{loc,0}(\widetilde{P}), \quad M \mapsto M - \Big\langle M, \int_0^\cdot \frac{\gamma\varphi}{1-\gamma}\, dW_s \Big\rangle$$

to $BMO(P)$ is a bijection between BMO(P)-martingales and BMO($\widetilde{P}$)-martingales. Consequently, $\int \widetilde{Z}\, dW$ is in $BMO(P)$ iff $\int \widetilde{Z}\, d\widetilde{W}$ is in $BMO(\widetilde{P})$ for $Z = (1-\gamma)\widetilde{Y}_-^{\gamma}\widetilde{Z}$ since $\Phi\Big(\int \widetilde{Z}\, dW\Big) = \int \widetilde{Z}\, dW - \int \frac{\gamma\varphi}{1-\gamma}\widetilde{Z}\, ds = \int \widetilde{Z}\, d\widetilde{W}$. In particular, $\widetilde{Z} \in \mathcal{H}^2$ iff $\widetilde{Z} \in \mathcal{H}^2(\widetilde{P})$. $\square$

According to Corollary 1.21 there exists a unique solution $(\widetilde{Y}, \widetilde{Z}, \widetilde{U}) \in \mathcal{S}^\infty(\widetilde{P}) \times \mathcal{H}^2(\widetilde{P}) \times \mathcal{H}^2_\nu(\widetilde{P})$ with positive Y-component to the BSDE (1.30) with

$$c^{\frac{1}{1-\gamma}} \exp\left(-\frac{\gamma|\varphi|^2_\infty}{2(1-\gamma)^2}(T-t)\right) \le \widetilde{Y}_t \le |\xi|_\infty \exp\left(\frac{\gamma|\varphi|^2_\infty}{2(1-\gamma)^2}(T-t)\right)$$

such that $\int \widetilde{Z}\, d\widetilde{W}$ and $\widetilde{U} * \widetilde{\mu}^{\widetilde{P}}$ are BMO($\widetilde{P}$)-martingales. By Lemma 1.36 and the statement of Proposition 1.35 that every bounded solution to the BSDE (ξ, f) is bounded from below away from zero in $Y \ge c > 0$, there is a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ with $\int Z\, dW \in BMO(P)$ and it is given by the coordinate transform (1.29). We note that Y (resp. $\widetilde{Y}$) can be interpreted as (dual) opportunity process, see [Nut10, Section 4], and summarize in

Theorem 1.37. *Assume $\lambda(E) < \infty$ and $k = d$. Let $f_s(y, z, u) = \frac{y}{2}\frac{\gamma}{1-\gamma}\left|\varphi_s + z/y\right|^2$ and let $\xi \in L^\infty(\mathcal{F}_T)$ with $\xi \ge c$ for some $c > 0$. Then there exists a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ with $\int Z\, dW \in BMO(P)$ to the BSDE (ξ, f). Moreover the strategy given by $\pi^*_s = \frac{1}{1-\gamma}\left(\varphi_s + \frac{Z_s}{Y_{s-}}\right)$ is optimal for the control problem (1.28), achieving the value $v_t(x) = \frac{x^\gamma}{\gamma} Y_t = V^{\pi^*}_t$.*

2. Hedging under generalized good-deal bounds in jump models with random measures

In this chapter, we study good-deal valuation and hedging under abstract discontinuous filtrations supporting a purely-discontinuous local-martingale random measure. For generalized no-good-deal restrictions on Girsanov kernels of pricing measures described by abstract correspondences, we derive good-deal bounds and associated hedging strategies in terms of solutions to JBSDEs studied in Chapter 1. This is partly the content of Section 2.2 which also includes some constructive examples for concrete no-good-deal restrictions (e.g. on Sharpe ratios or optimal growth-rates) and random measures (e.g. of semi-Markov processes or continuous time Markov chains). The Appendix 2.5 provides some intermediary results and proofs that are omitted form the main body of the chapter. First, let us provide the general mathematical and financial setup for this chapter, including some preliminaries about random measures and change of measures under discontinuous filtrations. For further terminologies of stochastic analysis not explained in this chapter, we refer to [JS03] and [HWY92].

2.1 Mathematical framework and preliminaries

The general setup is analogous to that of Chapter 1. Inequalities between random variables are understood P-a.e., and for processes will be understood $P \otimes dt$-a.e.. By the usual conditions on $\mathbb{F}$ all semimartingales are taken to be càdlàg and equality between two semimartingales implies indistinguishability. The financial market consists of d risky assets (stocks) and a riskless numéraire (bond) with unit price corresponding to a risk-free asset with zero interest rate. Hence all wealth processes are expressed in discounted units. The objective real world measure is P and we assume that under P the stock price process $S = (S^i)_{i=1,\ldots,d}$ is a positive locally bounded (càdlàg) semimartingale. We denote by $\mathcal{M}^e := \mathcal{M}^e(S)$ the set of equivalent local martingale measures for S and suppose that the financial market is arbitrage-free in the sense that $\mathcal{M}^e \neq \emptyset$. This assumption is indeed equivalent to the no free lunch with vanishing risk condition of [DS94]. For notational simplicity, we will often identify a measure $Q \ll P$ with its density process (alternatively the terminal value thereof in $L^1(P)$) $\Gamma^Q = dQ/dP$ with respect to P.

We endow the general filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ with an n-dimensional Brownian motion W (with $n \geq d$) and an integer-valued random measure μ given on the measurable space $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{E})$ by $\mu(dt, de) = \{\mu(\omega, dt, de),\ \omega \in \Omega\}$, where $E := \mathbb{R}^n \backslash \{0\}$

and $\mathcal{E} := \mathcal{B}(E)$. The predictable P-compensator of μ is denoted $\nu = \nu^P$. As usual, we denote $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$ the σ-field of functional processes defined on $\widetilde{\Omega} := \Omega \times [0,T] \times E$, and $\widetilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{E}$. Let $P \otimes \nu$ denote the positive measure defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ by $P \otimes \nu(\widetilde{B}) = E\big[\int_{[0,T]\times E} \mathbb{1}_{\widetilde{B}}(t,e)\nu(dt,de)\big]$, for all $\widetilde{B} \in \widetilde{\mathcal{F}}$. Inequality between $\tilde{\mathcal{P}}$-measurable function-valued processes will be understood in the $P \otimes \nu$-almost sure sense. For a measure $Q \ll P$, we will denote in the sequel ν^Q the compensator of μ under Q. Under Q, the compensated random measure $\widetilde{\mu}^Q := \mu - \nu^Q$ is by definition a purely discontinuous local martingale random measure. As in Chapter 1, we suppose that ν admits a density ζ with respect to a measure $\lambda \otimes dt$, such that $\nu(\omega, dt, de) = \zeta_t(\omega, e)\lambda(de) \otimes dt$, where λ is a σ-finite measure on $(E, \mathcal{E})$ such that $\int_E 1 \wedge |e|^2 \, \lambda(de) < \infty$ and ζ is a $\widetilde{\mathcal{P}}$-measurable function satisfying $0 \le \zeta \le c_\nu < \infty,\ P \otimes \lambda \otimes dt$-a.s. for a constant $c_\nu > 0$. Inequalities between $\mathcal{E}$-measurable functions will be understood in the λ-almost sure sense. The Absolute continuity of ν with respect to $\lambda \otimes dt$ implies that

$$\nu(\{0\} \times E) = \nu(\{t\} \times E) = 0,\ t \in [0,T] \tag{2.1}$$

and $\nu([0,T] \times E) \le c_\nu T \lambda(E),\ P\text{-a.s.}$. For all $t \in [0,T]$, we denote by λ_t the random measure defined by $\lambda_t(\omega)(de) := \zeta_t(\omega, e)\lambda(de)$.

The integral $U * \mu_t$ is defined as in Section 1.1 of Chapter 1. Analogously we define the predictable integral process $U * \nu$, and $E\left[|U| * \mu_T\right] = E\left[|U| * \nu_T\right]$ holds by the definition of the compensator. Again, results in [JS03, Section II.1] (applied for $\widehat{U} = 0$ since $\nu \ll \lambda \otimes dt$) imply that if $(|U|^2 * \mu)^{1/2}$ is locally integrable, then U is $\widetilde{\mu}$-integrable and $U * \widetilde{\mu}$ is defined as the purely discontinuous local P-martingale with jump process $(\int_E U_t(e)\ \mu(\{t\}, de))_{t\in[0,T]}$. In particular when $U \ge -1$, this holds if and only if the predictable increasing process $(1 - \sqrt{1+U})^2 * \nu$ is locally integrable. Moreover, U is $\widetilde{\mu}$-integrable and $U * \tilde{\mu}$ is a locally square integrable P-martingale (resp. square integrable P-martingale) if and only if the predictable increasing process $|U|^2 * \nu$ is locally integrable (resp. integrable). In this case, the predictable quadratic variation of $U * \tilde{\mu}$ is $\langle U * \widetilde{\mu} \rangle = |U|^2 * \nu$. If $|U| * \nu$ is locally integrable, then U is $\widetilde{\mu}$-integrable and $U * \widetilde{\mu} = U * \mu - U * \nu$.

For a matrix $A \in \mathbb{R}^{n\times d}$, A^{tr} denotes the transpose of A and $|A| := \big(\text{Trace } AA^{\mathrm{tr}}\big)^{1/2}$ its Euclidean norm in $\mathbb{R}^{n\times d}$. For a process $Y : [0,T] \times \Omega \to \mathcal{Y}$ with state space $\mathcal{Y}$, we sometimes write Y_t for $t \in [0,T]$ instead of $Y(t,\omega)$, when the dependence in $\omega \in \Omega$ is clear. The spaces $L^2(\lambda)$ and $L^2(\lambda_t) := L^2(\zeta_t d\lambda)$ are defined as in Section 1.1. Recall that $L^2(\lambda),\ L^2(\lambda_t(\omega)),\ (t,\omega) \in [0,T] \times \Omega$ are separable Hilbert spaces and therefore admit countable orthonormal bases. In addition for a probability measure Q, the spaces $L^p(Q),\ \mathcal{S}^p(Q),\ \mathcal{H}^2(Q),\ \mathcal{H}^2_\nu(Q)$ (for $1 \le p \le \infty$) are also defined analogously as in Section 1.1 of Chapter 1. We denote $\mathcal{M}_{loc}(Q)$ (resp. $\mathcal{M}^2(Q)$) the space of local (resp. square integrable) Q-martingales. Again for $Q = P$, we will simply write L^p, $\mathcal{S}^p$, $\mathcal{H}^2$, $\mathcal{H}^2_\nu$, $\mathcal{M}_{loc}$ and $\mathcal{M}^2$ for the spaces above. Recall that with these notations it holds that $Z \cdot W \in \mathcal{M}^2$ for $Z \in \mathcal{H}^2$, and

$U * \widetilde{\mu} \in \mathcal{M}^2$ for $U \in \mathcal{H}^2_\nu$. Integrands that produce identical stochastic integrals (almost-surely) will be identified in the same equivalence class.

Analogous to (1.2), a standing assumption for the sequel is the following weak predictable representation property of local martingales with respect to W, $\tilde{\mu}$ and the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$\text{For any } M \in \mathcal{M}_{loc}, \text{there exists } Z, U \text{predictable s.t. } M = M_0 + Z \cdot W + U * \widetilde{\mu}, \tag{2.2}$$

where $\int_0^T |Z|^2 dt < \infty$ and U is $\widetilde{\mu}$-integrable. Note that (2.2) implies (1.2). Indeed if the representation in (2.2) holds for a square integrable martingale M, both integrands must be at least locally square integrable and $\langle M \rangle = \int |Z|^2 \, dt + |U|^2 * \nu$ by strong orthogonality of the stochastic integrals. As a consequence $E[\langle M \rangle_T] < \infty$ would imply that Z and U are in the respective $\mathcal{H}^2$-spaces. For more about predictable representation properties for semimartingales see [HWY92, Chapter XIII.2] or [JS03, Chapter III, Section 4c]. For examples of situations where such a weak predictable representation property holds, we refer again to Example 1.1 of Chapter 1.

Assumption (2.2) will be needed in the sequel for application of the wellposedness results for JBSDE as in Chapter 1. Beyond this, it will be used for instance to give explicit representations of densities of no-good-deal measures as stochastic exponentials of integrals with respect to W and $\widetilde{\mu}$. The respective integrands ("Girsanov kernels") will be characterized as selections of some correspondence stemming from the economic interpretation of the no-good-deal restriction. To this end, the Girsanov theorem for general semimartingales (see [JS03, Theorem III.3.24]) plays a crucial role by providing a representation of characteristics of semimartingales under a change of measures. The latter will be obtained with respect to a specific truncation function $h : \mathbb{R}^n \to \mathbb{R}^n$, i.e. a bounded function with compact support s.t. $h(e) = e$ in a neighborhood of the origin. Typically h will be the canonical truncation function defined by $h(e) := e\mathbb{1}_{\{|e| \leq 1\}}$. Applying the Girsanov theorem, with respect to the canonical truncation function, to the semimartingale $\tilde{X}$ with P-characteristics $(B, c, \nu) := (0, I, \nu)$, where I is the $n \times n$ identity matrix gives the following lemma. The first claim of the lemma is a result of [JS03, Chapter III, Section 3.d], and the standard proof of the second claim is relegated to the appendix.

Lemma 2.1. *a) For $Q \ll P$ (resp. $Q \sim P$), let $\tilde{X}$ be the n-dimensional semimartingale with P-characteristics $(0, I, \nu^P)$ with respect to the canonical truncation function. Then there exists a $\widetilde{\mathcal{P}}$-measurable function $\gamma \geq -1$ (resp. $\gamma > -1$) and a predictable process β satisfying $|h\gamma| * \nu_t + \int_0^t |\beta_s|^2 \, ds < \infty$, Q-a.s. $t \in [0, T]$, such that a version (B^Q, c^Q, ν^Q) of the Q-characteristics of $\tilde{X}$ relative to h is given by $B^Q = \int_0^\cdot \beta_s ds + h\gamma * \nu$, $c^Q = I$ and $\nu^Q = (1 + \gamma) \cdot \nu$.*

b) If $Q \sim P$, then density process $\Gamma := dQ/dP$ is described as $\Gamma = \mathcal{E}(M)$, for a local martingale $M = \beta \cdot W + \gamma * \widetilde{\mu}$ with predictable integrands β, γ.

Remark 2.2. *In part a) of Lemma 2.1, the process γ expresses the change in the law of the jumps of the semimartingale $\tilde{X}$ with respect to a change of measure from P to Q. The process β intervenes together with γ in the change of drift of $\tilde{X}$, and the matrix c specifies the quadratic variation of $\tilde{X}$, which remains unchanged since it is invariant under an absolutely continuous change of measures. In particular $W^Q = W - \int_0^\cdot \beta_t dt$ is a Q-Brownian motion and the Girsanov transform of $\widetilde{\mu}$ under Q is the purely discontinuous local Q-martingale random measure $\widetilde{\mu}^Q = \mu - (1+\gamma)\nu = \widetilde{\mu} - \gamma\nu$.*

In the sequel, we will refer to the couple of integrands (γ, β) of Lemma 2.1 as the Girsanov kernels (identifying integrands that yield the same integral) of the measure $Q \ll P$ with respect to P and $\tilde{X}$, or more specifically with respect to P, W and $\widetilde{\mu}$. A reciprocal of part b) of Lemma 2.1 is stated analogously to [Klö06, Proposition 4.3.9] in Proposition 2.3 below; the proof in [Klö06] is easily adapted to our current setup beyond Lévy processes, and for completeness we include it in Appendix 2.5. This proof relies on Theorem II.5 of [LM78], restated as Proposition 2.31 in Appendix 2.5. Note that for $\Gamma = \mathcal{E}(M)$ with $M = \beta \cdot W + \gamma * \widetilde{\mu}$ one has

$$-\log(\Gamma) = -M + \frac{1}{2}\langle M^c \rangle - \sum_{s \le \cdot} \left(\log(1+\Delta M_s) - \Delta M_s\right) = -M + \frac{1}{2}\langle M^c \rangle + g(1+\gamma) * \mu,$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$g(y) := -\log y + y - 1. \tag{2.3}$$

The following Proposition 2.3 will be used to characterize in Section 2.4.2 the set of no-good-deal measures resulting from a no-good-deal constraint imposed as a bound on the conditional expected growth rates on investments in the financial market. The corresponding Girsanov kernels of no-good-deal measures will be described by conditions similar to (2.4) and (2.5) below. We will consider generalized no-good-deal constraints for which the no-good-deal pricing measures are risk neutral measures $Q^{\gamma,\beta}$ with Girsanov kernels (γ, β) that are selections of some correspondence with values contained in $\{\gamma \in L^2(\lambda) : \ \gamma > -1\} \times \mathbb{R}^n$.

Proposition 2.3. *Let g be the function defined in (2.3), β a predictable process and $\gamma > -1$ a $\widetilde{\mathcal{P}}$-measurable function. If there exists a constant $K > 0$ such that*

$$\int_E g(1+\gamma_t(e))\lambda_t(de) \le K, \quad t \in [0,T], \tag{2.4}$$

then γ is integrable with respect to $\widetilde{\mu}$. If in addition

$$|\beta_t|^2 \le K, \quad t \in [0,T], \tag{2.5}$$

then $\Gamma = \mathcal{E}(M)$, for $M = \beta \cdot W + \gamma * \widetilde{\mu}$, is a positive uniformly integrable P-martingale. In particular Γ is the density process of a measure $Q \sim P$ with Girsanov kernels (γ, β).

We will extensively use the theory of JBSDEs as exposed for instance in Chapter 1, to describe the good-deal bounds and their hedging strategies in our jump framework. Lemma 2.4 below is a re-statement of Lemma 1.23 in Chapter 1, for easy of reference to the convenience of the reader. As usual, it gives a representation result for the solution of a linear JBSDE in terms of the conditional expectation of the terminal value under a suitably parametrized probability measure. Note that this lemma and the upcoming Proposition 2.6 also hold for $\lambda(E)=\infty$.

Lemma 2.4. *Let f be of the linear form $f_t(y,z,u) := \alpha^0_t+\alpha_t y+\beta_t z+\int_E \gamma_t(e)u(e)\,\zeta(t,e)\,\lambda(de)$, for predictable α^0, α and β, and $\widetilde{\mathcal{P}}$-measurable γ. Let also $X\in L^\infty(P)$. Then*

1. *if $(Y,Z,U)\in\mathcal{S}^\infty\times\mathcal{H}^2\times\mathcal{H}^2_\nu$ solves the JBSDE with parameters (f,X) and the adjoint process $(\Gamma^t_s)_{t\le s\le T} := (\exp(\int_t^s \alpha_u\,du)\mathcal{E}(\beta\cdot W+\gamma*\tilde{\mu})^s_t)_{t\le s\le T}$ is in $\mathcal{S}^1$, $\forall t\in[0,T]$, with α^0 bounded, then Y has the representation*
$$Y_t=\mathbb{E}_t\big[\Gamma^t_T X+\int_t^T \Gamma^t_s\alpha^0_s\,ds\big],\quad t\in[0,T]. \tag{2.6}$$
2. *If α^0, α, β and $\tilde{\gamma}:=\int_E |\gamma_\cdot(e)|^2\,\zeta(\cdot,e)\lambda(de)$, are bounded for $\gamma\ge -1$, then there exists a unique solution (Y,Z,U) in $\mathcal{S}^\infty\times\mathcal{H}^2\times\mathcal{H}^2_\nu$ to the JBSDE with parameters (f,X) and hence (2.6) holds.*

Remark 2.5. *Let us note that a variant of part 1. of Lemma 2.4 holds for X being in L^2 with (Y,Z,U) in $\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2_\nu$ if the adjoint process Γ is in $\mathcal{S}^2$. Analogously, part 2. can be extended for $X\in L^2$, for the JBSDE solution (Y,Z,U) then being in $\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2_\nu$.*

It is known that a comparison theorem for BSDEs with jumps does not hold solely under classical hypotheses as in the pure Brownian case. Indeed, an additional monotonicity condition is needed to ensure comparison of the solutions of the JBSDEs from the comparison of the generators and terminal conditions. The following proposition provides a comparison theorem for JBSDEs stated as in Proposition 1.4 in Chapter 1, but with additional conditions that the Y-components of JBSDE solutions to compare are in $\mathcal{S}^2$ instead of $\mathcal{S}^\infty$, and the stochastic exponential $\mathcal{E}(\beta\cdot W+\overline{\gamma}*\tilde{\mu})$ is in $\mathcal{S}^2$. The proof being analogous to that of Proposition 1.4, it is omitted.

Proposition 2.6. *Let $(Y^i,Z^i,U^i)\in\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2_\nu$ be solutions to the JBSDE with parameters (f_i,X_i), $i=1,2$. Assume that the generator f_1 is Lipschitz continuous with respect to y and z. Let $\gamma:\Omega\times[0,T]\times\mathbb{R}^{n+3}\times E\to[-1,\infty)$ with $(\omega,t,y,z,u,u',e)\mapsto\gamma^{y,z,u,u'}_t(e)$ be a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{n+3})\otimes\mathcal{E}$-measurable function such that for $\overline{\gamma}:=\gamma^{Y^1_-,Z^1,U^1,U^2}$ it holds*
$$f_1(t,Y^1_t,Z^1_t,U^1_t)-f_1(t,Y^1_t,Z^1_t,U^2_t)\le\int_E\overline{\gamma}_t(e)\,(U^1_t(e)-U^2_t(e))\lambda_t(de),\ P\otimes dt\text{-a.s.} \tag{2.7}$$

and the stochastic exponential $\mathcal{E}(\beta \cdot W + \overline{\gamma} * \tilde{\mu})$ is a martingale in $\mathcal{S}^2$ for

$$\beta_t := \mathbb{1}_{\{Z_t^1 \neq Z_t^2\}} \frac{f_1(t, Y_t^1, Z_t^1, U_t^2) - f_1(t, Y_t^1, Z_t^2, U_t^2)}{\|Z_t^1 - Z_t^2\|^2}(Z_t^1 - Z_t^2), \quad t \in [0, T]. \tag{2.8}$$

Then $X_1 \leq X_2$ and $f_1(t, Y_t^2, Z_t^2, U_t^2) \leq f_2(t, Y_t^2, Z_t^2, U_t^2)$ imply $Y_t^1 \leq Y_t^2$ for all $t \leq T$.

Remark 2.7. *Reversing the roles of f_1 and f_2 in Proposition 2.6 with f_2 being Lipschitz continuous in y and z rather yields: for $X_1 \leq X_2$ and $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$, with (2.7) replaced by*

$$f_2(t, Y_t^2, Z_t^2, U_t^1) - f_2(t, Y_t^2, Z_t^2, U_t^2) \leq \int_E \overline{\gamma}_t(e)(U_t^1(e) - U_t^2(e))\lambda_t(de), \ P \otimes dt\text{-a.s.}$$

and β_t in (2.8) by

$$\beta_t = \mathbb{1}_{\{Z_t^1 \neq Z_t^2\}}(f_2(t, Y_t^2, Z_t^1, U_t^1) - f_2(t, Y_t^2, Z_t^2, U_t^1))\|Z_t^1 - Z_t^2\|^{-2}(Z_t^1 - Z_t^2)$$

with $\overline{\gamma} := \gamma^{Y_-^2, Z^2, U^1, U^2}$ and the stochastic exponential $\mathcal{E}(\beta \cdot W + \overline{\gamma} * \tilde{\mu})$ being a martingale in $\mathcal{S}^2$, it follows that $Y_t^1 \leq Y_t^2, \ t \in [0, T]$

2.2 Good-deal valuation and hedging

As already mentioned in the preliminaries, our financial market consists of d risky assets and a riskless bond with unit price. For our results on good-deal valuation and hedging, we model the risky asset prices $(S^i)_{i=1}^d =: S$ as (not necessarily Markovian) Itô processes solution of the SDE

$$dS_t = \operatorname{diag}(S_t)\sigma_t (\xi_t dt + dW_t) := \operatorname{diag}(S_t)\sigma_t d\widehat{W}_t, \ t \in [0, T], \quad S_0 \in (0, \infty)^d, \tag{2.9}$$

where σ is a predictable $\mathbb{R}^{d \times n}$-valued process of maximal rank $d \leq n$ (i.e. $\det(\sigma_t \sigma_t^*) \neq 0 \ P \otimes dt$-a.s.) and ξ is a predictable process in $\mathbb{R}^n$ with $\xi_t \in \operatorname{Im} \sigma_t^{\mathrm{tr}}, \ t \in [0, T]$. We assume that the market price of risk ξ is bounded. Boundedness of ξ ensures existence of the minimal martingale measure $\widehat{Q} \in \mathcal{M}^e \neq \emptyset$ with Girsanov kernels $(\widehat{\gamma}, \widehat{\beta}) := (0, -\xi)$. Moreover the compensators of μ under $\widehat{Q}$ and P coincide (up to indistinguishability), i.e. $\nu^{\widehat{Q}} = \nu$.

We describe the trading strategies in terms of amounts $(\varphi^i)_{i=1}^d =: \varphi$ of wealth invested in the risky assets with prices $(S^i)_{i=1}^d$. By a self-financing requirement, any pair (V_0, φ) yields a wealth process V with initial capital V_0 and satisfying $V = V_0 + (\sigma^{\mathrm{tr}} \varphi) \cdot \widehat{W}$. We define permitted trading strategies as couples (V_0, φ), with φ predictable and satisfying $E\left[\int_0^T |\varphi_t^{\mathrm{tr}} \sigma_t|^2 dt\right] < \infty$. Clearly trading in a permitted manner excludes doubling strategies and does not allow for arbitrage gains. Re-parameterizing permitted trading strategies in terms of integrands $\phi = \varphi^{\mathrm{tr}} \sigma$

with respect to $\widehat{W}$ yields wealth process dynamics $V = V_0 + \phi \cdot \widehat{W}$. There is a one-to-one relation between φ and ϕ given by $\phi = \sigma^{\mathrm{tr}}\varphi$ and $\varphi = (\sigma^{\mathrm{tr}})^{+}\phi$, where $(\sigma^{\mathrm{tr}})^{+} := (\sigma\sigma^{\mathrm{tr}})^{-1}\sigma$ is the pseudo-inverse of the matrix σ^{tr}. Denote by $\operatorname{Im}\sigma_t^{\mathrm{tr}}$ and $\operatorname{Ker}\sigma_t$ respectively the range and the kernel of the matrices σ_t^{tr} and σ_t. Since $(\operatorname{Im}\sigma_t^{\mathrm{tr}})^{\perp} = \operatorname{Ker}\sigma_t$, then every $z \in \mathbb{R}^n$ can be uniquely decomposed in terms of orthogonal projections $\Pi_t(\cdot)$ and $\Pi_t^{\perp}(\cdot)$ onto $\operatorname{Im}\sigma_t^{\mathrm{tr}}$ and $\operatorname{Ker}\sigma_t$ respectively as $z = \Pi_t(z) \oplus \Pi_t^{\perp}(z)$, $t \in [0,T]$. We define the set Φ of re-parameterized permitted trading strategies ϕ therefore as the set

$$\Phi := \left\{ \phi \;\middle|\; \phi \text{ is predictable, } \phi_t \in \operatorname{Im}\sigma_t^{\mathrm{tr}} \text{ and } E\left[\int_0^T |\phi_t|^2 dt\right] < \infty \right\}$$

.

By standard arguments, the equivalent local martingale measures can be characterized in terms of specific orthogonal decompositions of their Girsanov kernels as follows.

Lemma 2.8. *$\mathcal{M}^e$ consists of probability measures $Q \sim P$ with Girsanov kernels (γ, β), where β satisfies $\beta = -\xi + \eta$ with $\eta_t \in \operatorname{Ker}\sigma_t,\ t \in [0,T]$.*

The financial market model is typically incomplete since (by Lemma 2.8) it admits infinitely many equivalent local martingale measures, under which the associated compensators of the random measure μ may differ. Intuitively, incompleteness can also be seen by the fact that there is less tradeable risky assets than driving Brownian motions in the market (if $d < n$). Further, it could also be seen by the presence of unpredictable event-risk from the total inaccessibility of the jump times of some local-martingales. The latter can be seen as justifying the presence of intrinsic market risk, since there exists a purely discontinuous local Q-martingale, for some $Q \in \mathcal{M}^e$, which cannot be represented by an integral with respect to S, since S is continuous.

For a generalized notion of good-deal bounds in the jump setting, we consider correspondences C defined on $[0,T]\times\Omega$ and with values $C_t(\omega)$ subsets of $L^2(\lambda_t(\omega))\times\mathbb{R}^n$ such that Girsanov kernels (γ,β) of no-good-deal pricing measures are selections of C, i.e. $(\gamma_t(\omega), \beta_t(\omega)) \in C(t,\omega)$ (shortly written $(\gamma,\beta) \in C$). Such generalized good-deal bounds defined from abstract correspondences will also be considered in Chapter 3 in the Brownian filtration setting, where the γ-component of the Girsanov kernels will then be absent. Considering abstract correspondences provides a general framework for incorporating more concrete classical no-good-deal constraints. For instance, good-deal bounds from a constraint on the instantaneous Sharpe ratios [BS06] correspond to a radial correspondence C, one whose values $C(t,\omega)$ are closed balls in the Hilbert space $L^2(\lambda_t(\omega)) \times \mathbb{R}^n$. For $t \in [0,T]$ fixed, C_t can be seen as a correspondence defined on Ω, and we will sometimes write $(\gamma_t, \beta_t) \in C_t$ to mean $(\gamma_t(\omega), \beta_t(\omega)) \in C_t(\omega)$, for all $\omega \in \Omega$. We consider convex-valued correspondences C with values satisfying

$$C_t(\omega) \subseteq \Big\{(\gamma,\beta) \in L^2(\lambda_t(\omega)) \times \mathbb{R}^n :\ \gamma > -1\Big\},\ (t,\omega) \in [0,T]\times\Omega. \tag{2.10}$$

For a correspondence C satisfying (2.10) we will associate a closed-convex-valued correspondence $\widetilde{C}$ with values

$$\widetilde{C}_t(\omega) := \Big\{(\gamma,\beta) \in L^2(\lambda)\times\mathbb{R}^n :\ \big(\frac{\gamma}{\sqrt{\zeta_t(\omega)}}\mathbb{1}_{\{\zeta_t(\omega)>0\}},\beta\big) \in \bar{C}_t(\omega)\Big\}, \tag{2.11}$$

in the Hilbert space $L^2(\lambda)\times\mathbb{R}^n$, where $\bar{C}_t(\omega)$ is the closure of $C_t(\omega)$ in $L^2(\lambda_t(\omega))\times\mathbb{R}^n$, $(t,\omega)\in[0,T]\times\Omega$. Let $\overline{\mathcal{P}}$ denote the completion of the predictable σ-field $\mathcal{P}$ under the measure $P\otimes dt$. For the application of standard measurable selection arguments in our general setting, we will need the following

Assumption 2.9. *The correspondence $\widetilde{C}$ associated to C by (2.11) is $\overline{\mathcal{P}}$-measurable.*

Let us note at first that Assumption 2.9 will be satisfied (e.g. in Lemma 2.22 and Lemma 2.30) for some concrete examples of no-good-deal constraint correspondences and general measures λ. Also note that assuming $\overline{\mathcal{P}}$-measurability is weaker than assuming predictability. The definition of measurability of a correspondence is in the sense of [AF90]: for each closed set $F\subset L^2(\lambda)\times\mathbb{R}^n$, the set $\widetilde{C}^{-1}(F) := \Big\{(t,\omega)\in[0,T]\times\Omega :\ \widetilde{C}_t(\omega)\cap F\neq\emptyset\Big\}$ is measurable. Note that completeness of the underlying σ-field is usually required in the theory of measurable correspondences with values in an infinite-dimensional spaces. Since our correspondence $\widetilde{C}$ by definition assumes values in $L^2(\lambda)\times\mathbb{R}$ for a possibly infinitely-supported measure λ on $\mathbb{R}$, it appears natural to require Assumption 2.9 for $\overline{\mathcal{P}}$ instead of $\mathcal{P}$. For correspondences taking values in a finite dimensional space, completeness of the underlying σ-field is not necessary. For the theory of measurable correspondences and existence of measurable selection, we refer to [AF90, Chapter 8] in infinite dimensional spaces, and to [Roc76] in finite dimensional space. For measures λ that are finitely-supported (e.g. for finite state semi-Markov processes) we will therefore rely on results of [Roc76]. The particularity of $\widetilde{C}$ in comparison to C is that the range of $\widetilde{C}$ does not depend on (t,ω), and it will be useful for applying measurable selection arguments (e.g. in the proof of Lemma 2.14).

For predictable β and $\widetilde{\mathcal{P}}$-measurable γ such that $\Gamma^{\gamma,\beta} := \mathcal{E}\left(\beta\cdot W+\gamma*\widetilde{\mu}\right)$ is a positive uniformly integrable martingale, we denote by $Q^{\gamma,\beta}$ the probability measure equivalent to P with Girsanov kernels (γ,β), i.e. with density process $\Gamma^{\gamma,\beta}$. For risk-neutral measures $Q^{\gamma,\beta}\in\mathcal{M}^e$, the martingale condition of Lemma 2.8 additionally requires that $\beta=-\xi+\eta$ with $\eta_t(\omega)\in\operatorname{Ker}\sigma_t(\omega),\ (t,\omega)\in[0,T]\times\Omega$. Hence we define the set $\mathcal{Q}^{\rm ngd} := \mathcal{Q}^{\rm ngd}(S)$ of no-good-deal pricing measures as

$$\mathcal{Q}^{\rm ngd} := \Big\{Q^{\gamma,\beta}\sim P :\ (\gamma,\beta)\in C,\ \beta=-\xi+\eta,\ \eta\in\operatorname{Ker}\sigma\Big\}\subseteq\mathcal{M}^e, \tag{2.12}$$

where we do (implicitly) require that $\Gamma^{\gamma,\beta}$ is a uniformly integrable martingale to define probability measures $Q^{\gamma,\beta}$. Assume

$$(0,-\xi)\in C, \tag{2.13}$$

which implies in particular $\widehat{Q} = Q^{\widehat{\gamma},\widehat{\beta}} \in \mathcal{Q}^{\text{ngd}} \neq \emptyset$ since $(\widehat{\gamma},\widehat{\beta}) = (0,-\xi)$. In (2.12) we implicitly required the γ-components of Girsanov kernels to satisfy $\gamma_t \in L^2(\lambda_t)$ for any $t \in [0,T]$; see (2.10). Note that such a restriction is also made indirectly by [BS06], where the constraint on the size of the instantaneous Sharpe ratios via the Hansen-Jagannathan bounds implicitly requires the $L^2(\lambda_t)$-norm of γ_t to be finite, for any $t \in [0,T]$. For instance the latter holds if $\gamma * \widetilde{\mu}$ is a locally square integrable local martingale such that $|\gamma|^2 * \nu$ is locally integrable (cf. [JS03, Chapter II, Section 1]). Under uniform boundedness of the correspondence C (see Section 2.3), the local square integrability of $\gamma * \widetilde{\mu}$ will be automatically satisfied.

For sufficiently integrable contingent claims X, the upper and lower good-deal valuation bounds are defined by

$$\pi_t^u(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\text{ngd}}} E_t^Q[X] \quad \text{and} \quad \pi_t^l(X) := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^{\text{ngd}}} E_t^Q[X], \quad t \in [0,T]. \tag{2.14}$$

Because $\pi^l(X) = -\pi^u(-X)$, we focus on studying the upper good-deal bound. Recall the property of multiplicatively stable (shortly m-stable) sets $\mathcal{Q}$ of probability measures $Q \sim P$: for all $\Gamma^1, \Gamma^2 \in \mathcal{Q}$ and $\tau \le T$ stopping time, the process Γ with $\Gamma_t := \mathbb{1}_{\{t\le\tau\}}\Gamma_t^1 + \mathbb{1}_{\{\tau\le t\}}\Gamma_\tau^1\Gamma_t^2/\Gamma_\tau^2$ belongs to $\mathcal{Q}$, or equivalently the random variable $\Gamma_T := \Gamma_\tau^1\Gamma_T^2/\Gamma_\tau^2$ defines an element of $\mathcal{Q}$. The following result of [Del06] (stated here as in [KS07b, Theorem 2.7] or [Bec09, Proposition 2.6]) provides good dynamic properties for suprema of conditional expectations over an m-stable set of equivalent measures.

Lemma 2.10. *Let $\mathcal{Q}$ be a convex and m-stable set of measures $Q \sim P$ and $\pi_t^{u,\mathcal{Q}}(X) :=$ ess $\sup_{Q\in\mathcal{Q}}$ $E_t^Q[X]$, $t \in [0,T]$, $X \in L^\infty$. Then for all $X \in L^\infty$ there exists a càdàg version Y of $\pi_\cdot^{u,\mathcal{Q}}(X)$ such that $Y_\tau =$ ess $\sup_{Q\in\mathcal{Q}} E_\tau^Q[X] =: \pi_\tau^{u,\mathcal{Q}}(X)$ for any stopping time $\tau \le T$. Moreover $\pi_\cdot^{u,\mathcal{Q}}(\cdot)$ satisfies the properties of a dynamic coherent risk measure. It is recursive and time consistent: For all $\sigma \le \tau \le T$ stopping times and for all $X_1, X_2 \in L^\infty$, $\pi_\sigma^{u,\mathcal{Q}}(\pi_\tau^{u,\mathcal{Q}}(X^1)) = \pi_\sigma^{u,\mathcal{Q}}(X^1)$ and $\pi_\tau^{u,\mathcal{Q}}(X^1) \ge \pi_\tau^{u,\mathcal{Q}}(X^2)$ implies $\pi_\sigma^{u,\mathcal{Q}}(X^1) \ge \pi_\sigma^{u,\mathcal{Q}}(X^2)$. Finally, a supermartingale property holds: for all $Q \in \mathcal{Q}$ and for all stopping times $\sigma \le \tau \le T$, $\pi_\sigma^{u,\mathcal{Q}}(X) \ge E_\sigma^Q[\pi_\tau^{u,\mathcal{Q}}(X)]$. In particular, $\pi_\cdot^{u,\mathcal{Q}}(X)$ is a supermartingale under any $Q \in \mathcal{Q}$.*

One can show that the sets $\mathcal{Q}^{\text{ngd}}$, $\mathcal{M}^e$ are convex and m-stable, enabling an application of Lemma 2.10 to the good-deal bound $\pi_\cdot^u(X) = \pi_\cdot^{u,\mathcal{Q}^{\text{ngd}}}(X)$. That $\mathcal{M}^e := \mathcal{M}^e(S)$ is m-stable and convex is a consequence of [Del06, Proposition 5]. The next result shows that $\mathcal{Q}^{\text{ngd}}$ as defined in (2.12) is also convex and m-stable; the proof is included in the appendix.

Lemma 2.11. *The set $\mathcal{Q}^{ngd}$ is convex and m-stable.*

We use a notion of good-deal hedging similar to the one in [Bec09] (see also Chapter 3), where a hedging strategy arises as the minimizer of a suitable a-priori dynamic risk measure of no-good-deal type, for which the minimal capital to make the position acceptable coincides with the good-deal valuation bound. The dynamic risk measure to be minimized over all permitted trading strategies is defined for sufficiently integrable contingent claims X, as

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\mathsf{ngd}}} E_t^Q[X], \quad t \in [0,T],$$

where $\mathcal{P}^{\mathsf{ngd}}$ is a set of a-priori valuation measures to be chosen so that, in the spirit of [BE09], the good-deal bound $\pi^u_\cdot(X)$ becomes the market consistent risk measure (valuation) for X that arises from no-good-deal hedging with respect to ρ. An investor holding a liability X and trading parallely in the market according to a permitted strategy $\phi \in \Phi$ would assign to her position at every time $t \in [0,T]$ the risk $\rho_t(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s)$. For optimal trading, she would like to use a strategy $\bar{\phi} \in \Phi$ that minimizes her risk at any time $t \in [0,T]$, in such a way that the minimal capital requirement to make her position ρ-acceptable is $\pi^u_t(X)$. This minimum yields the market-based risk measure (after optimal risk-sharing with the financial market) in the spirit of [BE09]. In other words, good-deal valuation should arise from good-deal hedging by minimizing the dynamic coherent risk measure ρ with respect to the family $\mathcal{P}^{\mathsf{ngd}}$ of a-priori measure as generalized scenarios (see [ADE$^+$07]). This yields a hedging notion that corresponds to good-deal valuation, in that the market -based risk measure turns out to be the good-deal valuation bound $\pi^u_\cdot(X)$. The investor's hedging problem therefore is to find $\bar{\phi} \in \Phi$ such that

$$\pi^u_t(X) = \rho_t\Big(X - \int_t^T \bar{\phi}_s^{\mathrm{tr}} d\widehat{W}_s\Big) = \operatorname*{ess\,inf}_{\phi\in\Phi} \rho_t\Big(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\Big), \ t \in [0,T]. \tag{2.15}$$

Since $0 \in \Phi$, then (2.15) necessarily requires $\pi^u_t(X) \le \rho_t(X), \ t \in [0,T]$, which in turns hints that $\mathcal{P}^{\mathsf{ngd}}$ should contain the smaller set $\mathcal{Q}^{\mathsf{ngd}}$. As in [Bec09] we choose $\mathcal{P}^{\mathsf{ngd}}$ as the set of probability measures equivalent to P, that are not necessarily martingale measures and satisfy the no-good-deal constraint with C, i.e. such that $\mathcal{Q}^{\mathsf{ngd}} = \mathcal{P}^{\mathsf{ngd}} \cap \mathcal{M}^e$. More precisely we define

$$\mathcal{P}^{\mathrm{ngd}} := \Big\{Q^{\gamma,\beta} \sim P : \ (\gamma,\beta) \in C\Big\}. \tag{2.16}$$

In a financial market with no risky asset, i.e. $S \equiv 1$, any probability measure is a martingale measure and consequently $\mathcal{P}^{\mathsf{ngd}}$ defined in (2.16) coincides with $\mathcal{Q}^{\mathsf{ngd}}(1)$. Hence $\mathcal{P}^{\mathsf{ngd}}$ inherits the m-stability and convexity of $\mathcal{Q}^{\mathsf{ngd}}(1)$, and thus $\rho_\cdot(\cdot)$ is satisfies the properties in Lemma 2.10: ρ is a dynamic coherent time-consistent risk measure.

For a contingent claim X, the tracking error $R_t^\phi(X)$ from hedging according to a strategy $\phi \in \Phi$, at time $t \in [0,T]$, is defined by the difference between the capital variations of the claim and the profit/loss from dynamic trading up to time t according to ϕ, i.e.

$$R_t^\phi(X) := \pi^u_t(X) - \pi^u_0(X) - \phi \cdot \widehat{W}_t.$$

Remark 2.12. *For a self-financing strategy $\phi \in \Phi$ replicating $X = x_0 + \int_0^T \phi^{tr} d\widehat{W}$, with $x_0 \in \mathbb{R}$, the tracking error vanishes, i.e. $R^\phi(X) = 0$. One says that a strategy is a mean-self-financing* *(like risk-minimizing strategies studied in [Sch01, Section 2], with $E_t^{\widehat{Q}}[X]$ taking the role of $\pi_t^u(X)$) if its tracking error it is a martingale (under P). We will show (see Theorem 2.19) that the tracking error of a good-deal hedging strategy $\bar{\phi}$ satisfying (2.15) is a supermartingale under all a-priori measures in $\mathcal{P}^{ngd}$. The result will therefore enable us to view $\bar{\phi}$ as being "at least mean-self-financing" under any $Q \in \mathcal{P}^{ngd}$. This will be seen as a* robustness *property of $\bar{\phi}$ with respect to the set of measures $\mathcal{P}^{ngd}$ interpreted as generalized scenarios (in the sense of [ADE⁺07]).*

For results on good-deal valuation and hedging, we shall distinguish two cases, namely the case where the constraint correspondence is uniformly bounded, and the case beyond uniform boundedness. For the first case we will obtain descriptions of good-deal bounds and hedging strategies in terms of solutions to Lipschitz JBSDEs. In the second case, Lipschitz JBSDE tools are not directly applicable, and we will resort to approximation arguments focusing only on valuation.

2.3 Case of uniformly bounded correspondences

We characterize $\pi^u(X)$ as solution to a JBSDE under the assumption that C is uniformly bounded, which ensures that the resulting JBSDE has a Lipschitz generator function. Indeed, the connection between dynamic coherent risk measures and BSDEs is quite known; cf. e.g. [Ros06, PR15]. We say that a correspondence C satisfying (2.10) is uniformly bounded if

Assumption 2.13. $\sup_{(t,\omega)} \sup_{(\gamma,\beta)\in C_t(\omega)} (\|\gamma\|^2_{L^2(\lambda_t(\omega))} + |\beta|^2) < +\infty$.

Under Assumption 2.13 one can show as in [QS13, Proposition 3.2] that $\Gamma \in \mathcal{S}^2$ for any Γ density processes of a measure in $\mathcal{P}^{\mathsf{ngd}}$. Hence for contingent claims $X \in L^2 \supset L^\infty$ that may be path-dependent, $\pi_t^u(X)$ and $\rho_t(X)$ are well-defined as essential suprema of almost surely finite-valued random variables, and one can check (also for $X \in L^2$) that an analogue of Lemma 2.10 still holds. For each no-good-deal measure $Q \in \mathcal{Q}^{\mathsf{ngd}}$, Lemma 2.4 describes $E^Q_\cdot[X]$ as the value process of a JBSDE with linear generator. Then using the comparison principle Proposition 2.6, one can describe $\pi^u_\cdot(X)$ (and likewise for $\rho_\cdot(X)$) as the value process of a JBSDE whose generator is the supremum of the linear ones. The following Lemma 2.14 (see Appendix 2.5 for its proof using Assumption 2.13) that the maximum is indeed attained. This yields (cf. Theorem 2.16) a worst-case measure under which the good-deal bound is attained. Obviously such a worst-case measure will usually lie in the L^1-closure of the set $\mathcal{Q}^{\mathsf{ngd}}$ of no-good-deal measures.

Lemma 2.14. *Let Assumptions 2.9 and 2.13 hold for C satisfying (2.10). Let $\bar{C}_t(\omega)$ denote the closure of $C_t(\omega)$ in $L^2(\lambda_t(\omega)) \times \mathbb{R}^n$, $(t,\omega) \in [0,T] \times \Omega$ and let $Z \in \mathcal{H}^2$ and $U \in \mathcal{H}^2_\nu$. Then* *a) there exist* $\bar{\eta} = \bar{\eta}(Z, U)$ *predictable and* $\bar{\gamma} = \bar{\gamma}(Z, U)$ $\widetilde{\mathcal{P}}$*-measurable such that for* $P \otimes dt$*-almost all* $(\omega, t) \in \Omega \times [0, T]$ *holds*

$$(\bar{\gamma}_t(\omega), \bar{\eta}_t(\omega)) \in \underset{(\gamma_t(\omega), \eta_t(\omega))}{\textit{argmax}} \ \eta_t^{tr}(\omega)\Pi^{\perp}_{(t,\omega)}(Z_t(\omega)) + \int_E U_t(\omega, e)\gamma_t(\omega, e)\lambda_t(\omega, de), \quad (2.17)$$

with the supremum taken over all $(\gamma_t(\omega), \eta_t(\omega))$ *with* $(\gamma_t(\omega), -\xi_t(\omega) + \eta_t(\omega)) \in \bar{C}_t(\omega)$ *and* $\eta_t(\omega) \in \operatorname{Ker}\sigma_t(\omega)$.

b) there exist $\tilde{\beta} = \tilde{\beta}(Z, U)$ *predictable and* $\tilde{\gamma} = \tilde{\gamma}(Z, U)$ $\widetilde{\mathcal{P}}$*-measurable such that for* $P \otimes dt$*-almost all* $(\omega, t) \in \Omega \times [0, T]$ *holds*

$$(\tilde{\gamma}_t(\omega), \tilde{\beta}_t(\omega)) \in \underset{(\gamma_t(\omega), \beta_t(\omega)) \in \bar{C}_t(\omega)}{\textit{argmax}} \ \beta_t^{tr}(\omega)Z_t(\omega) + \int_E U_t(\omega, e)\gamma_t(\omega, e)\lambda_t(\omega, de).$$

To $(\bar{\gamma}, \bar{\beta} := -\xi + \bar{\eta}) \in \bar{C}$ of Part a) of Lemma 2.14, we associate the probability measure $\bar{Q} := Q^{\bar{\gamma}, \bar{\beta}} \ll P$, which might not be equivalent to P since $\bar{\gamma}$ may take the value -1 on a non-negligible set. So $\bar{Q}$ is possibly not an equivalent local martingale measure, but we now show that it belongs to the $L^1(P)$-closure of $\mathcal{Q}^{\text{ngd}}$.

Lemma 2.15. *For* $Z \in \mathcal{H}^2$ *and* $U \in \mathcal{H}^2_\nu$*, let* $(\bar{\gamma}, \bar{\eta})$ *be as in Part a) of Lemma 2.14. Define the measures* $\bar{Q} = Q^{\bar{\gamma}, \bar{\beta}} \ll P$ *for* $\bar{\beta} := -\xi + \bar{\eta}$ *and* $Q^n := \frac{1}{n}\widehat{Q} + (1 - \frac{1}{n})\bar{Q}$*, for all* $n \in \mathbb{N}$*. Then* $(Q^n)_{n\in\mathbb{N}} \subset \mathcal{Q}^{ngd}$ *and* Q^n *converges to* $\bar{Q}$ *in* $L^1(P)$ *as* $n \to \infty$*. Consequently, it holds* $\pi^u_t(X) \geq E^{\bar{Q}}_t[X],\ \bar{Q}\text{-a.s.},\ t \in [0, T]$.

Proof. Let $n \in \mathbb{N}$. Clearly $Q^n \sim P$. Moreover $dQ^n/dP = Z^n := \frac{1}{n}\widehat{Z} + (1 - \frac{1}{n})\bar{Z}$ with $\widehat{Z} := d\widehat{Q}/dP = \mathcal{E}(-\xi \cdot W)$ and $\bar{Z} := d\bar{Q}/dP$. Itô formula then yields $Z^n = \mathcal{E}((-\xi + \eta^n) \cdot W + \gamma^n * \tilde{\mu})$ for $\eta^n = \alpha\bar{\eta}$ being predictable and $\gamma^n = \alpha\bar{\gamma}$ is $\widetilde{\mathcal{P}}$-measurable with $\alpha = (1 - \frac{1}{n})\bar{Z}/Z^n \in [0, 1)$ thanks to $\widehat{Z} > 0$. Therefore $\eta^n \in \operatorname{Ker}\sigma$ and $\gamma^n > -1$ due to $\bar{\gamma} \geq -1$. Hence $(\gamma^n, \eta^n) \in C$ and so $Q^n = Q^{\gamma^n, \eta^n}$ is in $\mathcal{Q}^{\text{ngd}}$. Convergence of Q^n to $\bar{Q}$ in $L^1(P)$ as $n \to \infty$ is straightforward by definition of Q^n and this implies $\pi^u_t(X) \geq E^{\bar{Q}}_t[X]$ for all $t \leq T$. □

For $X \in L^2$, consider the two JBSDEs:

$$\begin{aligned} Y_t = X &+ \int_t^T \Big((-\xi_s + \bar{\eta}_s)^{\text{tr}} Z_s + \int_E U_s(e)\bar{\gamma}_s(e)\lambda_s(de)\Big)ds \\ &- \int_t^T Z_s^{\text{tr}} dW_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(ds, de), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} Y_t = X &+ \int_t^T \Big(Z_s^{\text{tr}}\tilde{\beta}_s + \int_E U_s(e)\tilde{\gamma}_s(e)\lambda_s(de)\Big)ds \\ &- \int_t^T Z_s^{\text{tr}} dW_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(ds, de), \end{aligned} \tag{2.19}$$

with $(\bar{\gamma}, \bar{\eta}) = (\bar{\gamma}(Z,U), \bar{\eta}(Z,U))$ and $(\tilde{\gamma}, \tilde{\beta}) = (\tilde{\gamma}(Z,U), \tilde{\beta}(Z,U))$ being by Lemma 2.14. Then we have the following

Theorem 2.16. *Let Assumptions 2.9 and 2.13 hold for C satisfying (2.10). Let $X \in L^2$, and $(\bar{\gamma}, \bar{\eta})$ and $(\tilde{\gamma}, \tilde{\beta})$ be as in Lemma 2.14. Then*

a) the JBSDE (2.18) has a unique solution (Y, Z, U) in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ and there exists $\bar{Q} \ll P$ with density $d\bar{Q}/dP = \mathcal{E}((-\xi + \bar{\eta}) \cdot W + \bar{\gamma} * \tilde{\mu})$ such that $\pi^u_\cdot(X)$ satisfies

$$\pi^u_t(X) = \underset{Q \in \mathcal{Q}^{ngd}}{\text{ess sup}}\, E^Q_t[X] = Y_t = E^{\bar{Q}}_t[X], \quad \bar{Q}\text{-}a.s., t \in [0,T].$$

b) the JBSDE (2.19) has a unique solution $(\tilde{Y}, \tilde{Z}, \tilde{U})$ in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ and there exists $\widetilde{Q} \ll P$ with $d\widetilde{Q}/dP = \mathcal{E}(\tilde{\beta} \cdot W + \tilde{\gamma} * \tilde{\mu})$ such that

$$\rho_t(X) = \underset{Q \in \mathcal{P}^{ngd}}{\text{ess sup}}\, E^Q_t[X] = E^{\widetilde{Q}}_t[X] = Y_t, \ \widetilde{Q}\text{-}a.s., t \in [0,T].$$

For $X \in L^2$ and permitted trading strategies $\phi \in \Phi$, consider the JBSDE

$$\begin{aligned} Y_t = X + \int_t^T \Big(-\xi_s^{\text{tr}} \phi_s + (Z_s - \phi_s)^{\text{tr}} \tilde{\beta}_s(Z - \phi, U) + \int_E U_s(e) \tilde{\gamma}_s(Z - \phi, U)(e) \lambda_t(de) \Big) ds \\ - \int_t^T Z_s^{\text{tr}} dW_s - \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de), \end{aligned} \tag{2.20}$$

where for $\phi \in \Phi$ the processes $\tilde{\gamma}_\cdot(Z - \phi, U)$ and $\tilde{\beta}_\cdot(Z - \phi, U)$ are as in Part b) of Lemma 2.14. Then we have the following lemma, whose proof is deferred to Appendix 2.5.

Lemma 2.17. *Let Assumptions 2.9 and 2.13 hold for C satisfying (2.10). For $X \in L^2$ and $\phi \in \Phi$, the JBSDE (2.20) admits a unique solution $(Y^\phi, Z^\phi, U^\phi) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ that satisfies $Y^\phi_t = \rho_t\Big(X - \int_t^T \phi_s^{tr} d\widehat{W}_s\Big), \ t \in [0,T]$.*

Let f and f^ϕ (for $\phi \in \Phi$) denote respectively the generators of the JBSDEs (2.18) and (2.20), given for $z \in \mathbb{R}^n, \ u \in L^2(\lambda_t), \ t \in [0,T]$ as

$$f(t,z,u) = \underset{\substack{(\gamma,\beta) \in \bar{C}_t \\ \beta \in -\xi_t + \operatorname{Ker} \sigma_t}}{\text{ess sup}} \quad (\beta^{\text{tr}} z + \int_E u(e) \gamma(e) \lambda_t(de)) \quad \text{and}$$

$$f^\phi(t,z,u) = -\xi_t^{\text{tr}} \phi_t + \underset{(\gamma,\beta) \in \bar{C}_t}{\text{ess sup}} \, (\beta^{\text{tr}}(z - \phi_t) + \int_E u(e) \gamma(e) \lambda_t(de)).$$

The following lemma will be used in combination with the comparison theorem for JBSDEs to show existence of a good-deal hedging strategy $\bar{\phi}$ solution to the hedging problem (2.15). The proof is also deferred to the Appendix 2.5.

Lemma 2.18. *Let Assumptions 2.9 and 2.13 hold for C satisfying (2.10). Then for all $z \in \mathbb{R}^n$, $u \in L^2(\lambda_t)$, $t \in [0,T]$ holds*

$$f(t,z,u) = \operatorname*{ess\,inf}_{\phi\in\Phi} f^{\phi}(t,z,u). \tag{2.21}$$

To prove a general existence result for $\bar{\phi}$ solution to the hedging problem (2.15), we will require the additional condition on the abstract correspondence C that

$$\text{there exists } \epsilon \in (0,1) \text{ such that } \{0\} \times B_\epsilon(-\xi_t(\omega)) \subseteq C_t(\omega), \quad \text{for all } (t,\omega), \tag{2.22}$$

where $B_\epsilon(-\xi)$ denotes the (closed or open) ball in $\mathbb{R}^n$ centered at $-\xi$ with radius ϵ. Condition (2.22) implies in particular (2.13), and will be automatically satisfied for concrete no-good-deal constraint correspondences as in the frameworks of Section 2.3.1 and Section 2.4.2. In addition, (2.22) ensures coercivity of the generators of the JBSDEs (2.20) as functions of $\phi \in \Phi$, i.e. $f^\phi(t,z,u) \to +\infty$ as $|\phi| \to \infty$ for fixed (t,z,u). Following common arguments in variational analysis, (2.22) will be used in the proof of Theorem 2.19 below to deduce existence of a minimizing strategy $\bar{\phi}$ in (2.15). The precise result is the following, and shows moreover that hedging strategies $\bar{\phi}$ are *at least mean-self-financing* in the sense that their tracking errors satisfy a supermartingale property with respect to measures in $\mathcal{P}^{\mathsf{ngd}}$. The proof is postponed to Appendix 2.5.

Theorem 2.19. *Let Assumptions 2.9 and 2.13 hold for C satisfying (2.10) and (2.22). For $X \in L^2$, let (Y,Z,U), (Y^ϕ, Z^ϕ, U^ϕ) in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ (for $\phi \in \Phi$) be solutions to the JBSDEs (2.18), (2.20) respectively, with $(\bar{\gamma},\bar{\eta})$, $(\tilde{\gamma},\tilde{\eta})$ as in Lemma 2.14. Then there exists $\bar{\phi} := \bar{\phi}(X) \in \Phi$ satisfying*

$$f^{\bar{\phi}}(t,Z_t,U_t) = \operatorname*{ess\,inf}_{\phi\in\Phi} f^{\phi}(t,Z_t,U_t), \ t \in [0,T],$$

and for such $\bar{\phi}$ hold $Y_t = \operatorname*{ess\,inf}_{\phi\in\Phi} Y^\phi_t = Y^{\bar{\phi}}_t, \ t \in [0,T]$, and

$$\pi^u_t(X) = \operatorname*{ess\,inf}_{\phi\in\Phi} \rho_t\Big(X - \int_t^T \phi^{tr}_s d\widehat{W}_s\Big) = \rho_t\Big(X - \int_t^T \bar{\phi}^{tr}_s d\widehat{W}_s\Big) = Y_t, \quad t \in [0,T]. \tag{2.23}$$

Moreover the tracking error $R^{\bar{\phi}}(X)$ of $\bar{\phi}$ is a Q-supermartingale for any $Q \in \mathcal{P}^{\mathsf{ngd}}$ and a Q^*-martingale, for $Q^* = Q^{\gamma^*,\beta^*} \in \mathcal{P}^{\mathsf{ngd}}$ with $(\gamma^*,\beta^*) := (\tilde{\gamma}(Z-\bar{\phi},U), \tilde{\beta}(Z-\bar{\phi},U))$.

Remark 2.20.

1. *In accordance with Remark 2.12, Theorem 2.19 shows that good-deal hedging strategies are robust in the sense that they are at least mean-self-financing with respect to the set $\mathcal{P}^{\mathsf{ngd}}$ as generalized scenarios.*

2. *Note that Theorem 2.19 shows only existence of $\bar{\phi}$ and does not claim its uniqueness. The latter may depend on the contingent claim X into consideration. Independently*

of the contingent claim, uniqueness may be obtained for particular structures of the correspondence C. We will provide examples (see Section 2.3.2 and last example of Section 2.3.1), where uniqueness is ensured for any claim, and explicit expressions of $\bar{\phi}$ in terms of JBSDE solutions can be obtained.

If the values of the correspondence C can be decomposed as $C_t = C_t^{\gamma} \times C_t^{\beta}$ for $C_t^{\gamma} \subset L^2(\lambda_t)$ and $C_t^{\beta} \subset \mathbb{R}^n$, $t \in [0,T]$, i.e. the no-good-deal constraint decouples as a constraint on the unpredictable event-risk separated from a constraint on the market price of stock risk, then the hedging strategy $\bar{\phi}$ does not depend on the U-component of the solution to the JBSDE (2.18). The precise statement is summarized in the following corollary of Theorem 2.19.

Corollary 2.21. *Let the conditions of Theorem 2.19 hold and assume in addition that $C_t = C_t^{\gamma} \times C_t^{\beta}$ for $C_t^{\gamma} \subset L^2(\lambda_t)$ and $C_t^{\beta} \subset \mathbb{R}^n$, $t \in [0,T]$. For $X \in L^2$, let (Y,Z,U) be the solution to the JBSDE (2.18). Then a good-deal hedging strategy is given by $\bar{\phi} \in \Phi$ satisfying*

$$\bar{\phi}_t \in \underset{\phi \in \Phi}{\mathit{argmin}} \Big(-\xi_t^{\mathit{tr}} \phi_t + \underset{\beta_t \in \bar{C}_t^{\beta}}{\mathit{ess\ sup}}\, \beta_t^{\mathit{tr}} (Z_t - \phi_t) \Big), \ t \in [0,T].$$

2.3.1 Results for constraint on instantaneous Sharpe ratios (bounded case)

We consider a no-good-deal constraint emanating from a bound on the instantaneous Sharpe ratios of investments in the financial market extended by additional derivative price processes. Recall that this case was studied in [BS06] for a Markovian model of asset prices and additional factor processes exhibiting jumps. [BS06, Theorem 2.3] showed an extended form of the Hansen–Jagannathan (HJ) inequality in the sense that $|SR_t| \leq \|(\gamma_t, \beta_t)\|_{L^2(\lambda_t)\times\mathbb{R}^n}, \ t \in [0,T]$, for all (γ, β) Girsanov kernels of measures in $\mathcal{M}^e$, where SR_t denotes the instantaneous Sharpe ratio at time t. This meant that a bound on the instantaneous Sharpe ratios could be achieved through a bound on the norm of the Girsanov kernels of pricing measures and an application of the HJ inequality. Their no-good-deal constraint was then set as

$$\|(\gamma_t, \beta_t)\|^2_{L^2(\lambda_t)\times\mathbb{R}^n} := \|\gamma_t\|^2_{L^2(\lambda_t)} + |\beta_t|^2 \leq K^2, \quad t \in [0,T], \tag{2.24}$$

for some given $K \in (0,\infty)$, and they derived the good-deal bounds in terms of solutions to HJB PIDEs using dynamic programming techniques. Here we rather use JBSDEs to derive good-deal bounds in a non-necessarily Markovian model under a no-good-deal constraint of the type (2.24), but for more general K being a positive bounded predictable process. The associated constraint correspondence C is then

$$C_t = \left\{ (\gamma, \beta) \in L^2(\lambda_t) \times \mathbb{R}^n : \gamma > -1, \ \|\gamma\|^2_{L^2(\lambda_t)} + |\beta|^2 \leq K_t^2 \right\}. \tag{2.25}$$

Beyond the boundedness of ξ, we assume that for some $\varepsilon > 0$ holds

$$K_t > |\xi_t| + \varepsilon, \quad t \in [0,T], \tag{2.26}$$

so that $(0,-\xi) \in C$ and hence $\widehat{Q} \in \mathcal{Q}^{\text{ngd}} \neq \emptyset$. Since K is bounded and predictable, the no-good-deal restriction in this example fits well into the framework of the current Section 2.3 since C given by (2.25) is then convex-valued, and satisfies (2.10) and Assumption 2.13. Were K not uniformly bounded, then C might fail to satisfy Assumption 2.13, and the framework of the upcoming Section 2.4 would then prevail. Here the closed-valued correspondence $\bar{C}$ is given by

$$\bar{C}_t = \Big\{(\gamma,\beta) \in L^2(\lambda_t) \times \mathbb{R}^n : \gamma \geq -1,\ \|\gamma\|^2_{L^2(\lambda_t)} + |\beta|^2 \leq K_t^2\Big\}. \tag{2.27}$$

Indeed for arbitrary $(\gamma,\beta) \in \bar{C}_t(\omega)$, one chooses the sequence $(\gamma^k,\beta^k)_{k\in\mathbb{N}} \subset C_t(\omega)$ with $\gamma^k = \gamma \vee (-1+\frac{1}{k})$ and $\beta^k = \beta$, so that $\left|\gamma^k\right| \leq |\gamma|$ holds (since $\gamma \geq -1$). By dominated convergence, it then follows that (γ^k,β^k) converges to (γ,β) in $L^2(\lambda_t(\omega)) \times \mathbb{R}^n$. The correspondence $\widetilde{C}$ defined as in (2.11) is then given by

$$\widetilde{C}_t = \Big\{(\gamma,\beta) \in L^2(\lambda) \times \mathbb{R}^n : \gamma \geq -\zeta_t^{1/2},\ \|\gamma\|^2_{L^2(\lambda)} + |\beta|^2 \leq K_t^2\Big\}. \tag{2.28}$$

Applying the theory of measurable correspondences in [AF90], the following lemma shows that Assumption 2.9 holds for $\widetilde{C}$. The proof is relegated to Appendix 2.5.

Lemma 2.22. *The closed-convex-valued correspondence $\widetilde{C}$ given by (2.28) is $\overline{\mathcal{P}}$-measurable.*

Now one can apply part a) of Theorem 2.16 to obtain a description of $\pi^u_\cdot(X)$ and a worst-case no-good-deal measure for $X \in L^2$ in terms of the solution to the JBSDE (2.18). The precise result is the following

Theorem 2.23. *For $X \in L^2$, the JBSDE (2.18) with $(\bar{\gamma},\bar{\eta})$ from (2.17) has a unique solution (Y,Z,U) in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$. Moreover there exists $\bar{Q} \ll P$ in the L^1-closure of $\mathcal{Q}^{ngd}$ (in the sense of Lemma 2.15), with density $d\bar{Q}/dP = \mathcal{E}\left((-\xi+\bar{\eta})\cdot W + \bar{\gamma} * \tilde{\mu}\right)$ such that the good-deal bound $\pi^u_\cdot(X)$ satisfies*

$$\pi^u_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{ngd}} E^Q_t[X] = Y_t = E^{\bar{Q}}_t[X] \quad \textit{for all } t \leq T. \tag{2.29}$$

Sufficient conditions for explicit formulas for valuation and hedging

Let us still consider the no-good-deal constraint on the Sharpe ratio described in terms of the correspondence C in (2.25). We investigate conditions ensuring an explicit form of the maximizer $(\bar{\gamma}(Z,U),\bar{\eta}(Z,U))$ in the generator of the JBSDE (2.18), which in turn may also lead to an explicit formula for the good-deal hedging strategy. Note that the classical Kuhn-Tucker routine may not apply for the maximization problem in (2.17) for C in (2.25), due to the additional constraint $\{\gamma \geq -1\}$. If one considers the good-deal valuation problem without this constraint for JBSDEs, then can obtain using Kuhn-Tucker arguments an explicit maximizer for all $t \in [0,T]$ as

$$\bar{\gamma}_t = \frac{\left(K_t^2 - |\xi_t|^2\right)^{1/2}}{\left(\|U_t\|^2_{L^2(\lambda_t)} + \left|\Pi_t^{\perp}(Z_t)\right|^2\right)^{1/2}} U_t \quad \text{and} \quad \bar{\eta}_t = \frac{\left(K_t^2 - |\xi_t|^2\right)^{1/2}}{\left(\|U_t\|^2_{L^2(\lambda_t)} + \left|\Pi_t^{\perp}(Z_t)\right|^2\right)^{1/2}} \Pi_t^{\perp}(Z_t). \quad (2.30)$$

In general the relaxed Girsanov kernels in (2.30) do not induce a measure $Q^{\bar{\gamma},-\xi+\bar{\eta}}$ that is absolutely continuous with respect to P. In addition, $(\bar{\gamma},\bar{\eta})$ from (2.30) only give rise to a relaxed bound $\pi^{u,r}(X)$ which is clearly larger than $\pi^u(X)$, i.e. $\pi^{u,r}(X) \geq \pi^u(X)$, for any financial risk X since it is obtained by maximizing $E_t^Q[X]$ over a set of measure $\mathcal{Q}_r \supseteq \mathcal{Q}^{\text{ngd}}$ containing eventually signed measures. These facts were already analyzed in [BS06, Section 3.5 and 4.4], where similar relaxed good-deal bounds was studied using Hamilton-Jacobi-Bellman techniques. In terms of JBSDEs, we obtain here the following relaxed version of the JBSDE (2.18), with an explicit generator and value process $\pi_\cdot^{u,r}(X)$ (instead of $\pi_\cdot^u(X)$) by replacing $(\bar{\gamma},\bar{\eta})$ in (2.18) by the expressions in (2.30)):

$$\begin{aligned} Y_t = X + \int_t^T \Big(-\xi_s^{\text{tr}} \Pi_s(Z_s) + \left(K_s^2 - |\xi_s|^2\right)^{1/2} \left(\|U_s\|^2_{L^2(\lambda_s)} + |\Pi_s^{\perp}(Z_s)|^2\right)^{1/2} \Big) ds \\ - \int_t^T Z_s^{\text{tr}} dW_s - \int_t^T \int_E U_s(e) \widetilde{\mu}(ds,de). \end{aligned} \quad (2.31)$$

The JBSDE (2.31) has a Lipschitz generator, and hence (by e.g. [Bec06, Proposition 3.2]) admits a unique solution $(Y,Z,U) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$, for $X \in L^2$, with $\pi_\cdot^{u,r}(X) := Y$. Note that for a Sharpe ratio bound K conveniently chosen (for example small enough), the relaxed good-deal bound $\pi_\cdot^{u,r}(X)$ could still be lower than the upper no-arbitrage bound. However in general $\pi_\cdot^{u,r}(X)$ may not be a no-arbitrage price since $(\bar{\gamma}, -\xi+\bar{\eta})$ in (2.30) could define a signed measure because $\bar{\gamma} \geq -1$ may be violated.

If for a contingent claim $X \in L^2$ one can show that $U \geq 0$ for (Y,Z,U) solving the JBSDE (2.31), then $\bar{\gamma}$ in (2.30) satisfies $\bar{\gamma} \geq 0 > -1$ and the generators of the JBSDEs (2.18) and (2.31) coincide. This would clearly imply that $\pi_\cdot^u(X) = \pi_\cdot^{u,r}(X)$, and both are described by the solution of the JBSDE (2.31). In this case, it would even possible to obtain a closed-form expression of good-deal hedging strategies $\bar{\phi}$. Precisely, we have the following

Proposition 2.24. *Assume the Sharpe ratio constraint described by the correspondence C in (2.25). For $X \in L^2$, let (Y,Z,U) be the unique solution to the JBSDE (2.31). Then*

1. *If the good-deal bound and its relaxed version coincide, i.e. $\pi_\cdot^u(X) = \pi_\cdot^{u,r}(X)$, then a good-deal hedging strategy $\bar{\phi} \in \Phi$ is given by*

$$\bar{\phi}_t = \frac{\left(\|U_t\|^2_{L^2(\lambda_t)} + \left|\Pi_t^{\perp}(Z_t)\right|^2\right)^{1/2}}{\left(K_t^2 - |\xi_t|^2\right)^{1/2}} \xi_t + \Pi_t(Z_t),\ t \in [0,T]. \quad (2.32)$$

2. *If* $U \geq 0$*, then* $\pi_t^u(X) = \pi_t^{u,r}(X) = E_t^{\bar{Q}}[X] = Y_t,\ t \in [0,T]$*, where* $\bar{Q} = Q^{\bar{\gamma},-\xi+\bar{\eta}}$ *is in* $\mathcal{Q}^{ngd}$ *with* $(\bar{\gamma}_t, \bar{\eta}_t) = \left(K_t^2 - |\xi_t|^2\right)^{1/2}\left(\|U_t\|^2_{L^2(\lambda_t)} + \left|\Pi_t^{\perp}(Z_t)\right|^2\right)^{-1/2}\left(U_t, \Pi_t^{\perp}(Z_t)\right)$*, satisfying* $\bar{\gamma} \geq 0 > -1$*.*

Proof. The second claim follows from the preceding discussion. As for the first claim, if $\pi_\cdot^u(X) = \pi_\cdot^{u,r}(X)$ then we have for any $t \in [0,T]$

$$\begin{aligned}
f^{\bar{\phi}}(t, Z_t, U_t) &= -\xi_t^{\mathrm{tr}}\bar{\phi}_t + \operatorname*{ess\,sup}_{(\gamma,\beta)\in\bar{C}_t} \beta^{\mathrm{tr}}(Z_t - \bar{\phi}_t) + \int_E \gamma(e)U_t(e)\lambda_t(de) \\
&\leq -\xi_t^{\mathrm{tr}}\bar{\phi}_t + K_t\Big(\|U_t\|^2_{L^2(\lambda_t)} + \left|Z_t - \bar{\phi}_t\right|^2\Big)^{1/2} \\
&= -\xi_t^{\mathrm{tr}}\Pi_t(Z_t) + \left(K_t^2 - |\xi_t|^2\right)^{1/2}\Big(\|U_t\|^2_{L^2(\lambda_t)} + |\Pi_t^{\perp}(Z_t)|^2\Big)^{1/2} \\
&= f(t, Z_t, U_t).
\end{aligned}$$

By Lemma 2.18 it follows that $f^{\bar{\phi}}(t, Z_t, U_t) = \operatorname{ess\,inf}_{\phi\in\Phi} f^{\phi}(t, Z_t, U_t),\ t \in [0,T]$, and therefore applying Theorem 2.19 proves the required claim. □

A condition similar to $U \geq 0$ for obtaining an explicit generator of the JBSDE describing the good-deal bound was provided in [Del12]. However [Del12] focused on a financial/insurance market with a single traded risky asset modelled by a two dimensional Brownian motion (i.e. $d = 1, n = 2$), and with the random measure μ associated to jumps of a point-process with state-independent compensator of the form $\nu(dt) = \zeta_t dt$ for a predictable process $\zeta \geq 0$. Our result deals with fairly general jump processes and financial markets (with $d \leq n$).

A drawback of the condition $U \geq 0$ in part 2. of Proposition 2.24 which ensures equality between $\pi_\cdot^u(X)$ and $\pi_\cdot^{u,r}(X)$ is that it might not be straightforward to check in general, and also depends on the contingent claim X into consideration. Examples of such claims includes derivatives X that solely on diffusive risk and for which one would expect that $U = 0$. Also, if μ is the random measure of the jumps of a simple Poisson process and X a claim that pays nothing if the Poisson process does not jump before maturity and a unit if it does, then one U would be non-negative. Recall that for $\pi_\cdot^u(X)$ and $\pi_\cdot^{u,r}(X)$ to coincide, it is sufficient to provide conditions that guarantee that the process $\bar{\gamma}$ defined in (2.30) satisfies $\bar{\gamma} \geq -1$. If the support of the measure λ is finite, one could write a condition on K and λ that does not depend on the claim and, without further hypotheses on U, ensures that $\bar{\gamma} \geq -1$ holds. Note that in this case, the controls in the optimization problem in Lemma 2.14 would live in a finite dimensional space, simplifying considerably the problem. To explain what happens in this case, we provide an example in a semi-Markov jump setup that includes continuous time Markov chains.

Example for semi-Markov jump-dynamics: We consider a Markov renewal process as $(J_n, T_n)_{n\in\mathbb{N}_0}$, with random variables $(J_n)_n$ taking values in the finite state space $E = \{e_1, \ldots, e_m\} \subset \mathbb{R}^m$ (w.l.o.g. to fit into our setup from Section 2.1), for e_i for $i \in \mathcal{I} := \{1, \ldots, m\}$ denoting the canonical unit vectors, with $m \in \mathbb{N}$, and a non-decreasing sequence $(T_n)_n$, $T_0 = 0$, of $\mathcal{F}$-measurable $[0, T]$-valued random times modeling its renewal times. We assume that the process $(J_n, T_n)_n$ starts almost-surely at a pre-specified state and has a stochastic semi-Markov kernel $\mathbf{Q} = (Q^{ij})_{i,j\in\mathcal{I}}$ exogenously given such that $Q^{ij}_t = P[J_{n+1} = e_j,\ T_{n+1} - T_n \le t \mid J_n = e_i]$, $t \in [0, T]$, for all $n \in \mathbb{N}$. The process J defined by $J_t := J_{N_t}$, where $N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \le t\}}$ is the counting process associated to the jumps of $(J_n, T_n)_n$, is called the semi-Markov process associated to the Markov-renewal process. Note that $(J_n)_n$ is a Markov chain with transition Kernel $\mathbf{P} = (p^{ij})_{i,j\in\mathcal{I}}$ satisfying $p^{ij} = Q^{ij}_T$. It is well-known that the conditional independence relation $Q^{ij}_t = p^{ij} G^{ij}_t$ holds, where $G^{ij}_t = P[T_{n+1} - T_n \le t \,|\, J_n = e_i,\ J_{n+1} = e_j]$, for all $n \in \mathbb{N}$, is the conditional distribution function of the sojourn time $T_{n+1} - T_n$ from state e_i to state e_j. For unexplained notions in the theory of Markov renewal processes, semi-Markov processes and point processes in general, we refer to [Ç75, Bre81].

Let μ be the random measure of the jumps of the semi-Markov process J, identified with the family of counting processes $(N^{ij})_{i,j\in\mathcal{I},\, j\neq i}$ through $\mu([0, t) \times D) = \sum_{i,j\in\mathcal{I},\, e_j - e_i \in D} N^{i,j}_t$, where $N^{ij}_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \le t,\, J_n = e_i,\, J_{n+1} = e_j\}}$ is the number of jumps up to time t from state e_i to state e_j of J. Note that $N_t = \sum_{i,j\in\mathcal{I},\, j\neq i} N^{ij}_t$. Let us denote by $\tau_t := \sup\{s \in [0, t] : J_{t-u} = J_t,\ u \in [t - s, t]\}$ the time the process J has spent in its current state J_t. Then the compensator ν_{ij} of N^{ij} is given by $\nu_{ij}(dt) = \mathbb{1}_{\{J_{t-} = e_i\}} \alpha^{ij}_t(\tau_{t-})dt$, with jump intensities

$$\alpha^{ij}_t(u) = \lim_{h\searrow 0} \frac{P[J_{t+h} = e_j \mid J_t = e_i,\ \tau_t = u]}{h}, \quad u, t \in [0, T],\ i, j \in \mathcal{I},\ i \neq j, \tag{2.33}$$

where we set $\alpha_{ii} = -\sum_{j\neq i} \alpha_{ij},\ i \in \mathcal{I}$. The compensated jump process $\widetilde{N}^{ij}$ of N^{ij} is $\widetilde{N}^{ij}_t = N^{ij}_t - \int_0^t \lambda^{ij}_s ds$, where $\lambda^{ij}_t := \mathbb{1}_{\{J_{t-} = e_i\}} \alpha^{ij}_t(\tau_{t-}),\ t \in [0, T]$. We identify $\widetilde{\mu}$ with the family $(\widetilde{N}^{ij})_{i,j\in\mathcal{I},\, j\neq i}$ and assume that $(W, \widetilde{\mu})$ have the weak predictable representation property with respect to $(\mathbb{F}, P)$ in the sense that for every local $(\mathbb{F}, P)$-martingale M, there exists predictable processes $Z, U = (U^{ij})_{i,j\in\mathcal{I},\, j\neq i}$ with $\int_0^T |Z_s|^2 ds < \infty$ and $\sum_{i,j\in\mathcal{I},\, j\neq i} \int_0^T \lambda^{ij}_s (U^{ij}_s)^2 ds < \infty$ such that $M_t = \int_0^t Z_s dW_s + \sum_{i,j\in\mathcal{I},\, j\neq i} \int_0^t U^{ij}_s d\tilde{N}^{ij}_s,\ t \in [0, T]$.

For $(Z, U) \in \mathcal{H}^2 \times \mathcal{H}^2_\nu$, the expressions of $\bar{\gamma}$ and $\bar{\eta}$ from (2.30) for $\bar{\gamma} = (\bar{\gamma}^{ij})_{i\in\mathcal{I},j\neq i}$ and $U = (U^{ij})_{i\in\mathcal{I},j\neq i}$ suitably re-parametrized in terms of integrands with respect to $\widetilde{N}^{ij}$ become

$$\bar{\gamma}^{ij}_t = \left(K^2_t - |\xi_t|^2\right)^{1/2} \Big(|\Pi^\perp_t(Z_t)|^2 + \sum_{k,l\in\mathcal{I},\, l\neq k} \lambda^{kl}_t (U^{kl}_t)^2\Big)^{-1/2} U^{ij}_t, \quad \text{and}$$

$$\bar{\eta}_t = \left(K^2_t - |\xi_t|^2\right)^{1/2} \Big(|\Pi^\perp_t(Z_t)|^2 + \sum_{k,l\in\mathcal{I},\, l\neq k} \lambda^{kl}_t (U^{kl}_t)^2\Big)^{-1/2} \Pi^\perp_t(Z_t).$$

Assume that the process K is small enough such that the inequality

$$(K^2 - |\xi|^2)\mathbb{1}_{\{\lambda^{ij}\neq 0\}} \leq \lambda^{ij},\ P\otimes dt\text{-a.s., for all } i,j\in\mathcal{I},\ i\neq j, \tag{2.34}$$

holds. Since $U\in\mathcal{H}^2_\nu$, we can assume without loss of generality that $U^{ij}=0$ on $\{\lambda^{ij}=0\}$. With this, (2.34) implies that for any $i,j\in\mathcal{I}$, $i\neq j$, $t\in[0,T]$ hold

$$\begin{aligned}(K_t^2-|\xi_t|^2)^{1/2}U_t^{ij} \geq -(K_t^2-|\xi_t|^2)^{1/2}|U_t^{ij}| &\geq -(\lambda_t^{ij})^{1/2}|U_t^{ij}|\\ &\geq -\Big(\sum_{k,l\in\mathcal{I},\, l\neq k}\lambda_t^{kl}|U_t^{ij}|^2\Big)^{1/2}\\ &\geq -\Big(|\Pi_t^\perp(Z_t)|^2+\sum_{k,l\in\mathcal{I},\, l\neq k}\lambda_t^{kl}(U_t^{kl})^2\Big)^{1/2}.\end{aligned}$$

Hence $\bar\gamma^{ij}\geq -1$ for any $i,j\in\mathcal{I},\ j\neq i$, which in turn ensures $\pi_\cdot^u(X)=\pi_\cdot^{u,r}(X)=Y$ for $X\in L^2$, with (Y,Z,U) solution to the JBSDE (2.31) which now reads

$$\begin{aligned}Y_t = X + \int_t^T\Big(-\xi_s^{\text{tr}}\Pi_s(Z_s)+(K_s^2-|\xi_s|^2)^{1/2}\Big(|\Pi_s^\perp(Z_s)|^2+\sum_{i,j\in\mathcal{I},\, j\neq i}\lambda_t^{ij}(U_t^{ij})^2\Big)^{\frac{1}{2}}\Big)ds\\ -\int_t^T Z_s^{\text{tr}}dW_s - \sum_{i,j\in\mathcal{I},\, j\neq i}\int_t^T U_t^{ij}d\tilde N_t^{ij}.\end{aligned} \tag{2.35}$$

Note that (2.34) implies $\bar\gamma\geq -1$ and not $\bar\gamma > -1$ in general. While the latter holds when $U\geq 0$, already the former is sufficient to ensure that $\pi_\cdot^u(X)=\pi_\cdot^{u,r}(X)$. For good-deal hedging, applying part 1. of Proposition 2.24 therefore implies that a good-deal hedging strategy exists for any contingent claim $X\in L^2$ and is expressed as

$$\bar\phi_t(X) = \frac{\Big(\sum_{i,j\in\mathcal{I},\, j\neq i}\lambda_s^{ij}(U_s^{ij})^2+\Big|\Pi_t^\perp(Z_t)\Big|^2\Big)^{1/2}}{(K_t^2-|\xi_t|^2)^{1/2}}\,\xi_t+\Pi_t(Z_t),\ t\in[0,T].$$

We now provide conditions under which the BSDE (2.35) can be reduced to a system of ODEs, which may then be solved numerically forward in time. To this end, suppose that the semi-Markov process is a continuous time Markov chain. For $A=(\alpha^{ij})_{i,j\in\mathcal{I}}$ denoting the deterministic but time-dependent rate matrix of the chain J with entries

$$\alpha^{ij}(t)=\lim_{h\searrow 0}\frac{P[J_{t+h}=e_j\mid J_t=e_i]}{h}\geq 0,\ t\in[0,T],$$

for $i\neq j$, and $\sum_{j\in\mathcal{I}}\alpha^{ij}=0$ for $i\in\mathcal{I}$, it follows that $\lambda_t^{ij}:=\mathbb{1}_{\{J_{t-}=e_i\}}\alpha^{ij}(t),\ t\in[0,T]$. As in [CS12] and analogously to the boundedness assumption on density ζ in the general setup, we assume that the components of the rate matrix process A are uniformly bounded so that with positive probability the Markov chain does not change state on fixed compact time intervals. For this part, we assume that W and J are independent, and $\mathbb{F}$ is the augmentation of $\mathbb{F}^W\vee\mathbb{F}^J$.

Using J as a factor process, assume that the market price of risk ξ and Sharpe ratio bound K are deterministic functions of the chain, i.e. $\xi_t := \xi(t, J_{t-})$ and $K_t := K(t, J_{t-})$, $t \in [0,T]$, for $K : [0,T] \times \mathbb{R}^m \to (0,\infty)$ and $\xi : [0,T] \times \mathbb{R}^m \to \mathbb{R}^n$ measurable. For a contingent claim $X = G(J_T)$, depending solely on the final state of the Markov chain, with $G : \mathbb{R}^m \to \mathbb{R}$ measurable, the Z-component of the JBSDE (2.35) vanishes by independence of W and J. Therefore $\pi^u(X) = Y$ holds, for (Y,U) solution to the JBSDE

$$Y_t = G(J_T) + \int_t^T (K_s^2 - |\xi_s|^2)^{1/2}\Big(\sum_{i,j\in\mathcal{I}\, j\neq i} \lambda_s^{ij}(U_s^{ij})^2\Big)^{\frac{1}{2}} ds - \int_t^T \sum_{i,j\in\mathcal{I}\, j\neq i} U_s^{ij} d\widetilde{N}_s^{ij}. \tag{2.36}$$

Now the results of [CS12] imply that the solution (Y,U) to (2.36) is Markovian, i.e.

$$Y_t = u(t, J_t) \quad \text{and} \quad U_t^{ij} = u(t,e_j) - u(t,e_i), \quad i,j \in \mathcal{I}, \quad t \in [0,T], \tag{2.37}$$

for a deterministic function $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$. Furthermore, u is such that the associated column vector function of time $u(t) := (u^i(t))_{i\in\mathcal{I}} \in \mathbb{R}^m$, with $u^i(t) := u(t,e_i)$, $i \in \mathcal{I}$, solves the coupled system of ODEs

$$\frac{du^i}{dt} = -\big(\big|K^i(t)\big|^2 - \big|\xi^i(t)\big|^2\big)^{1/2}\Big(\sum_{j\in\mathcal{I}} \alpha^{ij}(t)\big(u^j(t) - u^i(t)\big)^2\Big)^{\frac{1}{2}} - \sum_{j\in\mathcal{I}} \alpha^{ij}(t)\big(u^j(t) - u^i(t)\big), \tag{2.38}$$

$i \in \mathcal{I}$, with terminal condition $u(T) = (G(e_i))_{i\in\mathcal{I}}$, where we use the notation $K^i(t) := K(t,e_i)$, $\xi^i(t) := \xi(t,e_i)$, $i \in \mathcal{I}$, $t \in [0,T]$. Hence the good-deal bound is the solution of a system of ODEs, which by reversing the time can be transformed into an initial value problem easily solved by numerical ODE solver.

Example with explicit formulas for stronger Sharpe ratio constraints: Instead of considering specific conditions on contingent claims, or on the underlying jump process, let us focus on the no-good-deal constraint itself by considering a stronger Sharpe ratio constraint. Indeed the good-deal hedging strategy can be obtained explicitly using Corollary 2.21 if the Sharpe ratio constraint in the correspondence defined in (2.25) is reinforced by requiring rather

$$\max\{\|\gamma\|_{L^2(\lambda_t)}, |\beta|\} \le K_t/\sqrt{2}, \ t \in [0,T]. \tag{2.39}$$

Recall that the Euclidian norm $|\cdot|_2$ and maximum norm $|\cdot|_\infty$ are equivalent in $\mathbb{R}^2$ with $|\cdot|_\infty \le |\cdot|_2 \le \sqrt{2}|\cdot|_\infty$. Noting this, the upper (resp. lower) good-deal bounds obtained from the constraint correspondence (2.25) can be estimated from below (resp. above) by those obtained from the stronger Sharpe ratio constraint (2.39). We generalize (2.39) by decoupling the no-good-deal constraint into

$$\|\gamma\|_{L^2(\lambda_t)} \le K_t^\gamma \quad \text{and} \quad \beta^{\mathrm{tr}} A_t \beta \le (K_t^\beta)^2, \ t \in [0,T], \tag{2.40}$$

where A is a predictable $\mathbb{R}^{n\times n}$-matrix-valued process with symmetric values which are elliptic uniformly in $(t,\omega) \in [0,T] \times \Omega$, and K^γ, K^β are positive bounded predictable processes satisfying $K^\beta > \sqrt{\xi^{\mathrm{tr}} A\xi} + \varepsilon$ for some $\varepsilon > 0$. Under this generalized version we obtain below a closed-form expression for a good-deal hedging strategy $\bar{\phi}$. The correspondence associated to (2.40) is given for $t \in [0,T]$ by $C_t = C_t^\gamma \times C_t^\beta$ with

$C_t^\gamma = \{u \in L^2(\lambda_t) : u > -1$ and $\|u\|_{L^2(\lambda_t)} \leq K_t^\gamma\}$ and $C_t^\beta = \{x \in \mathbb{R}^n :\ x^{\mathrm{tr}} A_t x \leq (K_t^\beta)^2\}$.

Assuming that $A_t^{-1}(\operatorname{Ker}\sigma_t) = \operatorname{Ker}\sigma_t$ and that $|\xi_t| < K_t^\beta\sqrt{\alpha_t'}$, where α_t' is the ellipticity constant of $A_t^{-1},\ t \in [0,T]$. It follows from Corollary 2.21 and the upcoming Theorem 3.17 in Chapter 3 that the unique good-deal hedging strategy $\bar{\phi}$ is given by

$$\bar{\phi}_t = \frac{\Big(\Pi_t^\perp(Z_t)^{\mathrm{tr}} A_t^{-1}\Pi_t^\perp(Z_t)\Big)^{1/2}}{\big((K_t^\beta)^2 - \xi_t^{\mathrm{tr}} A_t\xi_t\big)^{1/2}} A_t\xi_t + \Pi_t(Z_t), \quad t \in [0,T],$$

for (Y,Z,U) solving the JBSDE (2.18).

2.3.2 Results for ellipsoidal constraint and uncertainty about jump intensities

We are also concerned with good-deal valuation and robust hedging with respect to uncertainty about intensities of jumps in the market. Here the investor faces uncertainty about the market price of jump risk which translates into *Knightian* uncertainty (ambiguity) about the real world measure. We assume that from empirical/historical data, the investor has isolated a confidence region $\mathcal{R}$ of candidate reference measures (subjective priors) centered around a probability measure P and described by

$$\mathcal{R} := \Big\{P^\psi \sim P\ :\ dP^\psi = \mathcal{E}(\psi * \widetilde{\mu})dP \quad \text{with} \quad \underline{\psi} \leq \psi \leq \bar{\psi}\Big\}, \tag{2.41}$$

where $-1 < \underline{\psi} \leq 0 \leq \bar{\psi}$ are $\widetilde{\mathcal{P}}$-measurable functions satisfying

$$\exists K \in (0,\infty) \text{ s.t. } \int_E \big(|\underline{\psi}_t(e)|^2 + |\bar{\psi}_t(e)|^2\big)\lambda_t(de) \leq K, \quad t \in [0,T]. \tag{2.42}$$

Under each reference measure $P^\psi \in \mathcal{R}$, we impose an *ellipsoidal* no-good-deal constraint on the market price of diffusion risk and *zero* no-good-deal constraint on the market price of jump-risk. In other words, the no-good-deal restriction is only imposed on the β-component of the Girsanov kernels (γ,β) of pricing measures in terms of an ellipsoidal correspondence and the γ-component is set to zero. The resulting set $\mathcal{Q}^{\mathrm{ngd}}(P^\psi) \subseteq \mathcal{M}^e(S,P^\psi) = \mathcal{M}^e(S,P) =: \mathcal{M}^e$ of no-good-deal measures under P^ψ is

$$\mathcal{Q}^{\mathrm{ngd}}(P^\psi) := \Big\{Q^\beta \sim P^\psi :\ dQ^\beta = \mathcal{E}(\beta\cdot W)dP^\psi,\ \beta \in C^\beta,\ \beta \in -\xi + \operatorname{Ker}\sigma\Big\}, \tag{2.43}$$

where $C_t^\beta(\omega) = \Big\{x \in \mathbb{R}^n : x^{\mathrm{tr}} A_t(\omega)x \leq (K_t^\beta)^2\Big\}$, $(t,\omega) \in [0,T]\times\Omega$, with A being a predictable $\mathbb{R}^{n\times n}$-matrix-valued process with symmetric values that are elliptic and bounded (in operator norm) uniformly in (t,ω), and K^β is a positive bounded predictable process satisfying $K^\beta > \sqrt{\xi^{\mathrm{tr}} A \xi} + \varepsilon$ for some $\varepsilon > 0$. The radial case corresponds to $A \equiv \mathrm{Id}_{\mathbb{R}^n}$. As in Section 3.2.1 of Chapter 3, assume the separability condition $A_t^{-1}(\operatorname{Ker}\sigma_t) = \operatorname{Ker}\sigma_t$ and that $|\xi_t| < K_t^\beta \sqrt{\alpha_t'}$, where α_t' is the ellipticity constant of $A_t^{-1},\ t \in [0,T]$. In Chapter 3 we will deal with robustness with respect to uncertainty about the drift of traded assets in a Brownian setting, following a worst-case multi-prior approach to ambiguity as in [GS89, CE02]. Here we consider a similar approach for uncertainty about the intensity of the underlying jumps described by the priors $P^\psi \in \mathcal{R}$. A seller who seeks for robustness can charge the largest valuation bound over all priors, in order to compensate for the eventual misspecification of intensities of the jumps. In this respect, for contingent claims $X \in L^2$, the worst-case approach under uncertainty yields the good-deal bounds

$$\pi_t^u(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathsf{ngd}}} E_t^Q[X],\ t \in [0,T]. \tag{2.44}$$

where $\mathcal{Q}^{\mathsf{ngd}} := \bigcup_{\underline{\psi} \le \psi \le \bar{\psi}} \mathcal{Q}^{\mathsf{ngd}}(P^\psi)$. Clearly, one can rewrite

$$\pi_t^u(X) = \operatorname*{ess\,sup}_{\underline{\psi} \le \psi \le \bar{\psi}} \ \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathsf{ngd}}(P^\psi)} E_t^Q[X].$$

By Yor's formula, it is seen that

$$\mathcal{Q}^{\mathrm{ngd}} = \left\{ Q^{\psi,\beta} \sim P :\ (\psi,\beta) \in C,\ \beta \in -\xi + \operatorname{Ker}\sigma \right\}, \tag{2.45}$$

where $C = C^\gamma \times C^\beta$ with

$$C_t^\gamma(\omega) := \left\{ \psi \in L^2(\lambda_t(\omega)) : \underline{\psi}_t(\omega) \le \psi \le \bar{\psi}_t(\omega) \right\} \subseteq L^2(\lambda_t(\omega)),\ (t,\omega) \in [0,T] \times \Omega.$$

Hence $\mathcal{Q}^{\mathrm{ngd}}$ is m-stable and convex (cf. Lemma 2.11). By (2.42), uniform ellipticity of A and boundedness of K^β, the correspondence C satisfies Assumption 2.13. Moreover by standard measurable selection arguments, the associated closed-convex-valued correspondence $\widetilde{C}$ defined in (2.11) clearly satisfies Assumption 2.9. Hence the set $\mathcal{Q}^{\mathrm{ngd}}$ falls in the general framework of Section 2.2 for a set of no-good-deal measure defined as in (2.12) with the associated correspondence $C = C^\gamma \times C^\beta$ satisfying the uniform boundedness and measurability hypotheses of Theorem 2.16. Note that the main difference between the two constraints is that $C_t^\gamma(\omega)$ from (2.40) is given in terms of a L^2-bound on the integrands γ of the Girsanov kernels, whereas the current one is given in terms of pointwise bounds on the integrands γ.

For $Z \in \mathcal{H}^2$ and $U \in \mathcal{H}^2_\nu$, the optimal Girsanov kernels $(\bar{\gamma}, \bar{\eta})$ of part a) of Lemma 2.14, can be explicitly derived from the corresponding maximization problem (2.17) and for $t \in [0,T]$ as

$$\bar{\gamma}_t = \underline{\psi}_t \mathbb{1}_{\{U_t < 0\}} + \bar{\psi}_t \mathbb{1}_{\{U_t > 0\}} + 0 \mathbb{1}_{\{U_t = 0\}} \quad \text{and} \quad \bar{\eta}_t = \frac{\left((K_t^\beta)^2 - \xi_t^{\mathsf{tr}} A_t \xi_t \right)^{1/2}}{\left(\Pi_t^\perp(Z_t)^{\mathsf{tr}} A_t^{-1} \Pi_t^\perp(Z_t) \right)^{1/2}} A_t^{-1} \Pi_t^\perp(Z_t),$$

with $\bar{\gamma}$ clearly satisfying $\bar{\gamma} > -1$. Part a) of Theorem 2.16 now applies and the good-deal bound $\pi^u_\cdot(X)$, for $X \in L^2$, is described by $\pi^u_t(X) = Y_t = E^{\bar{Q}}_t[X]$, $t \in [0,T]$, for the worst-case no-good-deal measure $\bar{Q} = Q^{\bar{\gamma},-\xi+\bar{\eta}}$ in $\mathcal{Q}^{\mathsf{ngd}}$ and $(Y,Z,U) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ solution to the Lipschitz JBSDE (2.18) which in the present setup rewrites explicitly as

$$\begin{aligned} Y_t = X + \int_t^T \Big(&- \xi_s^{\mathsf{tr}} \Pi_s(Z_s) + \Big((K_s^\beta)^2 - \xi_s^{\mathsf{tr}} A_s \xi_s \Big)^{1/2} \Big(\Pi_s^\perp(Z_s)^{\mathsf{tr}} A_s^{-1} \Pi_s^\perp(Z_s) \Big)^{1/2} \\ &+ \int_E \Big(\underline{\psi}_s(e) \mathbb{1}_{\{U_s(e)<0\}} + \bar{\psi}_s(e) \mathbb{1}_{\{U_s(e)>0\}} \Big) U_s(e) \lambda_s(de) \Big) ds \qquad (2.46) \\ - \int_t^T Z_s^{\mathsf{tr}} dW_s - \int_t^T \int_E U_s(e) \widetilde{\mu}(ds, de). \end{aligned}$$

To show that the correspondence $C = C^\gamma \times C^\beta$ satisfies (2.22), note at first that since A is uniformly bounded in the operator norm, there exists a constant $a \in (0, \infty)$ such that $\|A_t(\omega)\| \le a$ for a.a. (t,ω). Now if $|x + \xi_t| < \epsilon$ holds, then by the inequality $K^\beta > \sqrt{\xi^{\mathsf{tr}} A \xi} + \varepsilon$ one has

$$(x^{\mathsf{tr}} A_t x)^{1/2} \le \left((x+\xi_t)^{\mathsf{tr}} A_t (x+\xi_t) \right)^{1/2} + (\xi_t^{\mathsf{tr}} A_t \xi_t)^{1/2} < \|A_t\|^{1/2} |x+\xi_t| - \varepsilon + K_t^\beta < \epsilon a^{1/2} - \varepsilon + K_t^\beta.$$

Now choosing $\epsilon \in (0,1)$ such that $\epsilon \le \varepsilon a^{-1/2}$ implies that (2.22) holds. Hence the correspondence C satisfies the conditions of Corollary 2.21, which together with the results of Section 3.2.1 in Chapter 3 (cf. Theorem 3.17 therein) implies that the good-deal hedging strategy $\bar{\phi}(X)$ is uniquely given by

$$\bar{\phi}_t(X) = \frac{\Big(\Pi_t^\perp(Z_t)^{\mathsf{tr}} A_t^{-1} \Pi_t^\perp(Z_t) \Big)^{1/2}}{\left((K_t^\beta)^2 - \xi_t^{\mathsf{tr}} A_t \xi_t \right)^{1/2}} A_t \xi_t + \Pi_t(Z_t), \quad t \in [0,T],$$

for (Y,Z,U) being solution to the JBSDE (2.46). Note that since $\mathcal{P}^{\mathsf{ngd}} = \bigcup_{\underline{\psi} \le \psi \le \bar{\psi}} \mathcal{P}^{\mathsf{ngd}}(P^\psi)$ then, as expected, the good-deal hedging strategy $\bar{\phi}$ is also *robust with respect to uncertainty* in the sense that its tracking error $R^{\bar{\phi}}(X)$ satisfies a supermartingale property under all measures in $\mathcal{P}^{\mathsf{ngd}}(P^\psi)$ uniformly for all reference priors $P^\psi \in \mathcal{R}$. For similar results in the Brownian setting, we refer to Chapter 3 with uncertainty about market price of (diffusion) risk and to Chapter 4 with uncertainty about the volatility of tradeable assets.

2.4 Case of non-uniformly bounded correspondences

Beyond Assumption 2.13, let us now consider a convex-valued correspondence C that still satisfies (2.10) but may fail to be uniformly bounded. In this case the generator of the JBSDE (2.18) may not be Lipschitz continuous, and results on Lipschitz JBSDEs may not apply as in the case of uniformly bounded correspondences. However for a non-uniformly bounded correspondence C, one can still derive approximations of the good-deal bound $\pi^u_\cdot(X)$ by solutions to Lipschitz JBSDEs arising from truncations of the correspondence C which satisfy Assumption 2.13. Here the density processes Γ of no-good-deal measures may not be in $\mathcal{S}^2$ and $X \in L^2$ may no longer imply $X \in L^1(Q)$ for all $Q \in \mathcal{Q}^{\mathsf{ngd}}$. For this reason, we shall restrict the study here to financial claims $X \in L^\infty$. We consider a sequence $C^k_t := \{(\gamma, \beta) \in C_t : \ \|\gamma\|^2_{L^2(\lambda_t)} + |\beta|^2 \le k^2\},\ k \in \mathbb{N}$, of correspondences satisfying Assumption 2.13. Since C is convex-valued and satisfies (2.10), then each C^k is also convex-valued and satisfies (2.10). Moreover, since the correspondence $\widetilde{C}$ given as in (2.11) satisfies Assumption 2.9, one can show using arguments similar to those in the proof of Lemma 2.22 that the correspondences $\widetilde{C}^k_t = \left\{(\gamma, \beta) \in L^2(\lambda) \times \mathbb{R}^n : \ (\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma, \beta) \in \bar{C}^k_t\right\}$ are also $\overline{\mathcal{P}}$-measurable, where $\bar{C}^k_t(\omega)$ denotes the closure of $C^k_t(\omega)$ in $L^2(\lambda_t(\omega)) \times \mathbb{R}^n$. Since each $C^k,\ k \in \mathbb{N}$, satisfies Assumption 2.13, then results of Section 2.3 are applicable if one replaces C by any of the $C^k,\ k \in \mathbb{N}$. In addition $C^k_t(\omega) \nearrow C_t(\omega),\ \text{ as } k \nearrow \infty$. For $k \in \mathbb{N}$, denote $\mathcal{Q}^{\mathsf{ngd}}_k$ the set defined in (2.12) with C^k instead of C and consider the associated process

$$\pi^{u,k}_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^{\mathsf{ngd}}_k} E^Q_t[X],\ t \in [0, T].$$

The correspondences C^k can be interpreted as describing a no-good-deal constraint consisting of the initial constraint in C in addition to a constraint on the instantaneous Sharpe ratios given by the constant bound $K = k \in \mathbb{N}$. Note that the sets $\mathcal{Q}^{\mathsf{ngd}}_k,\ k \in \mathbb{N}$, also are convex and m-stable (by Lemma 2.11). First we have the following

Lemma 2.25. *Let $X \in L^\infty$. Then the following dynamic principles hold:*

1. *$\pi^u(X)$ is the smallest adapted càdlàg process such that it is a supermartingale under every $Q \in \mathcal{Q}^{\mathsf{ngd}}$ with terminal value X.*
2. *For all $k \in \mathbb{N}$, $\pi^{u,k}(X)$ is the smallest adapted càdlàg process such that it is a supermartingale under every $Q \in \mathcal{Q}^{\mathsf{ngd}}_k$ with terminal condition X.*

Proof. The supermartingale properties of $\pi^u(X)$ and $\pi^{u,k}(X)$ respectively under $Q \in \mathcal{Q}^{\mathsf{ngd}}$ and $Q \in \mathcal{Q}^{\mathsf{ngd}}_k$ are consequences of m-stability and convexity of $\mathcal{Q}^{\mathsf{ngd}}$ and $\mathcal{Q}^{\mathsf{ngd}}_k$ (see Lemma 2.10). That they are respectively the smallest ones follows from definitions as essential suprema of closed martingales of the type $E^Q_\cdot[X]$. □

For non-uniformly bounded correspondences, the following Theorem 2.26 describes in detail the approximation of good-deal valuation bounds for unbounded correspondences C by solutions to JBSDEs obtained from truncations C^k of C, which are uniformly bounded and fit to the setting of Section 2.3. This is an analogue of Theorem 3.7 for a possibly discontinuous filtration. Note however the presence of an additional part (part 5.) here. We mention that both theorems are only concerned with approximations for good-deal valuation bounds, and not with hedging strategies. It is an interesting question if the hedging strategies associated to the approximating bounds converge in some sense to a process that can somehow be interpreted as hedging strategy. We do not investigate further on this issue. The proof of Theorem 2.26 is postponed to Appendix 2.5.

Theorem 2.26. *Let C be a correspondence satisfying (2.10) and such that Assumption 2.9 holds. For a contingent claim $X \in L^\infty$, hold:*

1. $\pi_t^{u,k}(X) \nearrow \pi_t^u(X)$ *a.s. as* $k \to \infty$, *for all* $t \in [0,T]$
2. *For any* $k \in \mathbb{N}$, $\pi^{u,k}(X) = Y^k$ *for* $(Y^k, Z^k, U^k) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ *unique solution to the Lipschitz JBSDE*
$$\begin{aligned} Y_t = X + \int_t^T \Big(-\xi_s^{tr}\Pi_s(Z_s) + \operatorname*{ess\,sup}_{\substack{(\gamma_s,\eta_s)\in\bar{C}_s^k+(0,\xi_s)\\ \eta_s\in\operatorname{Ker}\sigma_s}} (\eta_s^{tr}\Pi_s^\perp(Z_s) + \int_E U_s(e)\gamma_s(e)\lambda_s(de))\Big)ds \\ - \int_t^T Z_s^{tr} dW_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(ds,de), \qquad (2.47) \end{aligned}$$
3. $\pi^u(X)$ *and* $\pi^{u,k}(X)$ *for* $k \geq \|\xi\|_\infty$ *admit under* $\widehat{Q}$ *the Doob-Meyer decompositions*
$$\pi^u(X) = \pi_0^u(X) + Z \cdot \widehat{W} + U * \widetilde{\mu} - A, \text{ and} \qquad (2.48)$$
$$\pi^{u,k}(X) = \pi_0^{u,k}(X) + Z^k \cdot \widehat{W} + U^k * \widetilde{\mu} - A^k, \qquad (2.49)$$
where $(Z,U) \in \mathcal{H}^2(\widehat{Q}) \times \mathcal{H}^2_\nu(\widehat{Q})$ *and* A, A^k *are non-decreasing predictable processes with* $A_T, A_T^k \in L^2(\widehat{Q}),\ A_0^k = A_0 = 0$, *and*
$$A^k = \int_0^\cdot \operatorname*{ess\,sup}_{\substack{(\gamma_t,\eta_t)\in\bar{C}_t^k+(0,\xi_t)\\ \eta_t\in\operatorname{Ker}\sigma_t}} \Big(\eta_t^{tr}\Pi_t^\perp(Z_t^k) + \int_E U_t^k(e)\gamma_t(e)\lambda_t(de)\Big)dt. \qquad (2.50)$$
4. *For all* $u \in [0,T]$, A_u^k *converges to* A_u *weakly in* $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$, Z^k *converges to* Z *weakly in* $L^2(\Omega \times [0,u], \widehat{Q} \otimes dt)$, *and* U^k *converges to* U *weakly in* $L^2(\Omega \times E \times [0,u], P \otimes \nu)$ *as* $k \to \infty$.
5. *If there exists* $\bar{Q}$ *in the* L^1*-closure of* $\mathcal{Q}^{\text{ngd}}$, *such that* $\pi_0^u(X) = E^{\bar{Q}}[X]$, *then* $\pi_\cdot^u(X)$ *is a quasi-left-continuous* $\bar{Q}$*-martingale, and* $\pi_\cdot^{u,k}(X)$ *converges to* $\pi_\cdot^u(X)$ *in* $\mathcal{S}^p(\widehat{Q})$ *for any* $p \in [1,\infty)$. *Moreover* $\big[(Z^k - Z) \cdot \widehat{W} + (U^k - U) * \widetilde{\mu}\big]_T^{1/2}$ *converges to* 0 *in* $L^1(\widehat{Q})$, *and* A^k *converges to* A *in* $\mathcal{S}^1(\widehat{Q})$ *with* $E^{\widehat{Q}}[A_T] \leq 2\|X\|_\infty$, *with* Z^k, Z, U^k, U *from 3..*

In Theorem 2.26, part 5., the hypothesis on existence of a worst-case measure $\bar{Q}$ for $\pi_\cdot^u(X)$ in the L^1-closure of $\mathcal{Q}^{\text{ngd}}$ is ensured for any contingent claim $X \in L^\infty$ if the set of densities Z_T^Q

(with respect to P) of measures Q in $\mathcal{Q}^{\mathrm{ngd}}$ is weakly relatively compact in L^1 (i.e. uniformly integrable, by Dunford-Pettis compactness theorem [DM78, Chapter II, Theorem 25]). This is a consequence of James' theorem (cf. [AB06, Theorem 6.36]). In case the correspondence C is uniformly bounded, existence of $\bar{Q}$ is proved in part a) of Theorem 2.16, and the approximations in Theorem 2.26 are not necessary in the first place. An example of a correspondence that may not satisfy Assumption 2.13 and for which the set of densities of measures in $\mathcal{Q}^{\mathrm{ngd}}$ is uniformly integrable is given by

$$C_t = \Big\{(\gamma,\beta) \in L^2(\lambda_t) \times \mathbb{R}^n : \ \gamma > -1 \quad \text{and} \quad \frac{1}{2}|\beta|^2 + \int_E \tilde{g}(1+\gamma(e))\lambda_t(de) \leq K\Big\}, \tag{2.51}$$

with $K \in (0,\infty)$ and the nonnegative function $\tilde{g}$ defined by $\tilde{g}(y) = y\log y - y + 1,\ y > 0$. Such a correspondence results from a no-good-deal constraint imposed as a dynamic bound $K^2(\sigma-\tau)$ on the *conditional relative entropy* $E_\tau\big[\frac{\Gamma_\sigma}{\Gamma_\tau}\log\frac{\Gamma_\sigma}{\Gamma_\tau}\big]$ for stopping times $\tau \leq \sigma \leq T$ and density processes Γ from $\mathcal{Q}^{\mathrm{ngd}}$ (see e.g. also [Klö06, Chapter 3]). For the correspondence C, uniform integrability of $\mathcal{Q}^{\mathrm{ngd}}$ is ensured by applying the de la Vallée Poussin's theorem [DM78, Chapter II, Theorem 22]. In lack of more general assumptions for the worst-case measure $\bar{Q}$ to exists, its existence could be checked in some specific situations, using the specific structure of the claim X in the model at hand. An example for this will be given in Section 3.2.2 of Chapter 3, where X is a put option in the Heston model, $\mathbb{F}$ is the augmented Brownian filtration and C is a radial correspondence modeling an unbounded constraint on the instantaneous Sharpe ratios (as e.g. in Section 2.4.1 below).

2.4.1 Results for constraint on instantaneous Sharpe ratios (unbounded case)

Consider good-deal bounds from a constraint on a Sharpe ratio described by the radial correspondence

$$C_t = \Big\{(\gamma,\beta) \in L^2(\lambda_t) \times \mathbb{R}^n : \ \gamma > -1,\ \|\gamma\|^2_{L^2(\lambda_t)} + |\beta|^2 \leq K_t^2\Big\}, \tag{2.52}$$

for a positive predictable process K that could unbounded. Similarly to Section 2.3.1, C is convex-valued, satisfies (2.10), and is such that Assumption 2.9 holds for the associated correspondence $\widetilde{C}$ defined as in (2.11). However C does not satisfy Assumption 2.13 if K is unbounded, since for $\beta_t := K|\beta^0|^{-1}\beta^0$, with $\beta^0 \in \mathbb{R}^n \setminus \{0\}$, one has $(0,\beta) \in C$ but $\sup_{(t,\omega)} |\beta_t(\omega)| \geq |K|_\infty = \infty$. Hence the correspondence C in (2.52) fits the setup of Section 2.4 and therefore the associated good-deal bounds can be described by Theorem 2.26.

2.4.2 Results for constraint on optimal expected growth rates

For another application, we consider good-deal bounds emanating from a constraint on the optimal expected growth rates of log-returns. The set $\mathcal{Q}^{\mathrm{ngd}}$ for such a constraint can be formulated in terms of a bound on the conditional reverse relative entropy of no-good-deal measures with respect to the reference measure P. Recall that for stopping times $\tau \leq \sigma$ and a measure Q equivalent to P with density process Γ, the $\mathcal{F}_\tau$-*conditional reverse relative entropy* $H^\sigma_\tau(P\,|\,Q)$ of Q with respect to P is the *conditional f-divergence* (for $f = -\log$) defined as $E_\tau\big[-\log\frac{\Gamma_\sigma}{\Gamma_\tau}\big] =: H^\sigma_\tau(P\,|\,Q) \geq 0$, with non-negativeness following from the P-submartingale property of $-\log\Gamma$ (cf. Proposition 2.27). [Klö06, KS07b] studied dual representations of (static) good-deal bounds and their dynamic properties in a Lévy framework with constraints on the f-divergence for diverse choices of the function f corresponding to logarithmic ($f(z) = -\log(z)$), exponential ($f(z) = z\log z$ corresponding to constraint (2.51) on the conditional relative entropy) and power ($f(z) = z^p,\ p \geq 1$) utility functions. Related pricing (and hedging) approaches from a constraint on generalized relative entropy are considered in [Lei08]. In [Bec09] it was shown in a Brownian setting that a dynamic bound on the reverse relative entropy of risk-neutral measures (in $\mathcal{M}^e$) corresponds to a bound on the optimal expected growth rates of (log-)returns, for any extension of the financial market by additional derivative price processes that are computed as conditional expectations under no-good-deal pricing measures (in $\mathcal{Q}^{\mathsf{ngd}}$). This provides a no-good-deal constraint that, in the Brownian setting, is essentially equivalent to the constraint on the instantaneous Sharpe ratios. We first note that for a discontinuous filtration (presence of jumps), the two no-good-deal constraints are no longer equivalent and the constraint on the Sharpe ratios as in Section 2.3.1 appears mathematically more tractable in terms of JBSDEs. In fact, the correspondence resulting from a constraint on optimal growth rates may fail to satisfy Assumption 2.13, even when the growth rates are bounded by a constant.

In this section, we derive good-deal bounds and show existence of hedging strategies for a no-good-deal constraint on optimal growth rates, using Lipschitz JBSDEs in a setup with jumps of finite state semi-Markov processes, i.e. in particular having a finitely-supported jump compensator. The restriction to finite state space is important as this ensures that the resulting JBSDEs have classical Lipschitz continuous generators and existence of good-deal hedging strategies can be shown as in Theorem 2.19. Beyond finitely supported compensators, the results in this section may not guarantee existence of a good-deal hedging strategy for such no-good-deal constraints since the associated correspondence C in (2.57) may be unbounded; yet we do still have result on good-deal valuation bounds. To be more precise in our current setup, let K be a positive bounded predictable process and define $\mathcal{Q}^{\mathsf{ngd}}$ as consisting of measures $Q \in \mathcal{M}^e$ that satisfy

$$H^\sigma_\tau(P\,|\,Q) \leq \frac{1}{2}E_\tau\Big[\int_\tau^\sigma K_u^2\,du\Big], \quad \text{for all } \tau \leq \sigma \leq T, \tag{2.53}$$

where τ, σ are stopping times. Let us recall [Bec09, Proposition 2.2] providing some useful properties of the process $-\log\Gamma$, for $Q \sim P$ with finite reverse entropy.

Proposition 2.27. *Let $Q \sim P$ with density process Γ of Q with respect to P such that* $\log\Gamma_T \in$

L^1. *Then* $-\log(\Gamma)$ *is a* P*-submartingale of class (D) with a Doob-Meyer decomposition* $-\log(\Gamma) = N + A$, *where* N *is a uniformly integrable* P*-martingale and* A *a predictable, non-decreasing and* P*-integrable process, with* $N_0 = A_0 = 0$. *Moreover* $E_\tau\big[-\log\frac{\Gamma_\sigma}{\Gamma_\tau}\big] = E_\tau\left[A_\sigma - A_\tau\right]$ *holds for all stopping times* $\tau \leq \sigma \leq T$.

Using Proposition 2.27, we can reformulate the definition of $\mathcal{Q}^{\mathrm{ngd}}$ in terms of deterministic times. To this end, define positive measures υ and $\kappa = \kappa^Q$ on the predictable σ-field $\mathcal{P}$ by $\upsilon(B) := \frac{1}{2}E\left[\int_0^T \mathbb{1}_B K_u^2 du\right]$ and $\kappa(B) := E\left[\int_0^T \mathbb{1}_B dA_u\right]$, $B \in \mathcal{P}$, where A is the non-decreasing process in the Doob-Meyer decomposition of $-\log(\Gamma)$, for Γ being the density process of a measure $Q \sim P$. We have the following equivalent condition for $Q \sim P$ to be in $\mathcal{Q}^{\mathrm{ngd}}$ (see [Bec09, Proposition 2.3]).

Proposition 2.28. *For* $Q \sim P$ *with density process* Γ *satisfying*

$$E_s\big[-\log\frac{\Gamma_t}{\Gamma_s}\big] \leq \frac{1}{2}E_s\big[\int_s^t K_u^2 du\big] \text{ for all deterministic times } s \leq t \leq T, \tag{2.54}$$

holds $\kappa(B) \leq \upsilon(B)$ *for any* $B \in \mathcal{P}$. *In particular, condition (2.54) is equivalent to its stopping time analogue (2.53). Thus a measure* $Q \in \mathcal{M}^e$ *is element of* $\mathcal{Q}^{\mathrm{ngd}}$ *if and only if (2.54) holds.*

One can interpret, as mentioned above, the constraint (2.53) as a bound on the optimal expected growth rates in the financial market extended by additional derivative price processes (see [Bec09, Theorem 3.1]). Using Lemma 2.8, one can formulate a definition of the set $\mathcal{Q}^{\mathrm{ngd}}$ by a condition on the Girsanov kernels of the associated measures. For $Q \in \mathcal{M}^e$, the following proposition derives N and A from Proposition 2.27 in our setup in terms of the Girsanov kernels of Q. The proof is deferred to Appendix 2.5.

Proposition 2.29. *Let* $Q \sim P$ *with Girsanov kernels* (γ, β) *and density process* $\Gamma = \mathcal{E}(M)$ *where* $M = \beta \cdot W + \gamma * \widetilde{\mu}$ *as in part b) of Lemma 2.1, and let also* g *be defined by* (2.3). *Then*

1. *If* $Q^{\gamma,\beta} \in \mathcal{M}^e$ *with* $E[-\log \Gamma_T^{\gamma,\beta}] < \infty$, *then* $\beta = -\xi + \eta$, *with* $\Pi_t^\perp(\beta_t) = \eta_t$, $t \in [0,T]$. *Moreover the Doob-Meyer decomposition* $-\log\Gamma^{\gamma,\beta} = N^{\gamma,\beta} + A^{\gamma,\beta}$ *is given by*

$$\begin{aligned} N^{\gamma,\beta} &= -\beta \cdot W - \log(1+\gamma) * \widetilde{\mu} \text{ and} \\ A^{\gamma,\beta} &= \int_0^\cdot \Big(\frac{1}{2}\left|\beta_s\right|^2 + \int_E g(1+\gamma_s(e))\lambda_s(de)\Big)ds. \end{aligned} \tag{2.55}$$

2. *If* $Q^{\gamma,\beta} \in \mathcal{Q}^{\mathrm{ngd}}$, *then* $\beta = -\xi + \eta$ *and*

$$\frac{1}{2}\left|\eta_t\right|^2 + \int_E g(1+\gamma_t(e))\lambda_t(de) \leq \frac{1}{2}K_t^2 - \frac{1}{2}\left|\xi_t\right|^2, \quad t \in [0,T]. \tag{2.56}$$

3. *Reciprocally, any $\widetilde{\mathcal{P}}$-measurable $\widetilde{\mu}$-integrable function $\gamma > -1$ and any predictable process β with $\Pi_t(\beta_t) = -\xi_t,\ t \in [0,T]$ satisfying $\frac{1}{2}|\beta_t|^2 + \int_E g(1+\gamma_t(e))\lambda_t(de) \leq \frac{1}{2}K_t^2,\ t \in [0,T]$, define a measure $Q^{\gamma,\beta} \in \mathcal{Q}^{\mathsf{ngd}}$ with Girsanov kernels $(\gamma, -\xi+\eta)$, where $\eta_t = \Pi_t^{\perp}(\beta_t),\ t \in [0,T]$.*

Assume that $K > |\xi| + \varepsilon$, for some $\varepsilon \in (0,1)$, ensuring that $\widehat{Q} \in \mathcal{Q}^{\mathsf{ngd}} \neq \emptyset$. From Proposition 2.29, the constraint correspondence satisfying (2.10) for the set of no-good-deal measures defined by (2.12) can be chosen as

$$C_t := \Big\{(\gamma,\beta) \in L^2(\lambda_t) \times \mathbb{R}^n :\ \gamma > -1, \quad \|2g(1+\gamma)\|_{L^1(\lambda_t)} + |\beta|^2 \leq K_t^2\Big\}. \tag{2.57}$$

The correspondence C has non-empty convex values (since $(0,0) \in C$ and g is convex). However, it is easily seen that C given by (2.57) does not satisfy Assumption 2.13 in general. Indeed, assume that μ is the random measure of the big jumps of a Gamma Lévy process with parameters $a = b = 1$, i.e. with $E = \mathbb{R} \setminus \{0\}$, $\zeta \equiv 1$ and $\lambda(dx) = \exp(-x)x^{-1}\mathbb{1}_{\{x\geq 1\}}dx$. Then for the function γ with $\gamma(x) = \exp(x/2)\mathbb{1}_{\{x\geq 1\}}$ satisfies $\int_E g(1+\gamma(x))\lambda(dx) < \infty$ and $\int_E |\gamma(x)|^2 \lambda(dx) = \infty$. Therefore for suitably chosen $K \in (0,\infty)$ the sequence $((\gamma^n, 0))_{n\in\mathbb{N}}$ of Girsanov kernels with $\gamma^n = \gamma\mathbb{1}_{[1,n]}$ is included in C but is not bounded in $L^2(\lambda)$. It is for this reason that we present this case as an application of Section 2.4 which works generally beyond Assumption 2.13.

Note that around $-1 < \gamma \leq 1$, one has the following Taylor approximation of g up to the (leading) second order: $g(1+\gamma) = -\log(1+\gamma) + \gamma = \frac{\gamma^2}{2} + O(\gamma^3)$. In this sense, the constraint on Sharpe ratios can be viewed as an approximation of that on optimal growth rates, for pricing measures $Q^{\gamma,\beta}$ possessing a low market prices of jump-risk γ. It is therefore not surprising that for continuous filtrations (absence of jump-risk, i.e. trivial $\mu = \nu = 0$), formally $\gamma = 0$ and the two types of no-good-deal constraints are equivalent; cf. [Bec09]. Clearly, a (bounded) constraint on the Sharpe ratios is mathematically more tractable since it naturally leads to standard Lipschitz JBSDEs for good-deal valuation and hedging.

The correspondence C in (2.57) has been defined so that it satisfies (2.10) but, for our theory, we will show that its associated correspondence $\widetilde{C}$ satisfies Assumption 2.9. First we describe the closure $\bar{C}_t$ of the set C_t in $L^2(\lambda_t) \times \mathbb{R}^n$. This needs some preparation because the function g and its derivative g' explode in the neighborhood of 0. Consider the pointwise approximation $(g^l)_{l\in\mathbb{N}}$ of g consisting of non-negative Lipschitz functions

$$g^l(y) := \begin{cases} g(\frac{1}{l}), & \text{if } 0 \leq y \leq \frac{1}{l} \\ g(y), & \text{if } y \geq \frac{1}{l}. \end{cases}$$

The sequence $(g^l)_l$ is non-decreasing and converges pointwise to g on $(0,\infty)$ as l tends to infinity. In particular for any $l \in \mathbb{N}$ the function g^l satisfies $g^l(1+y) \leq \text{Const}\,|y|^2$ for all $y \geq -1$ for some Const > 0. This property will be useful later in the proof of the second claim of

Lemma 2.30. Note that the function $g(1+\cdot)$ is dominated by Const $|y|^2$ only locally around the origin. Now define for each $l \in \mathbb{N}$ the correspondence

$$\bar{C}^l_t := \Big\{(\gamma,\beta) \in L^2(\lambda_t) \times \mathbb{R}^n : \ \gamma \geq -1, \ \|2g^l(1+\gamma)\|_{L^1(\lambda_t)} + |\beta|^2 \leq K_t^2\Big\}.$$

Since g^l is continuous and non-negative, then by Fatou's lemma the sets $\bar{C}^l_t$ are closed in $L^2(\lambda_t) \times \mathbb{R}^n$, for each $l \in \mathbb{N}, \ t \in [0,T]$. The related correspondence $\widetilde{C}^l$ according to (2.11) therefore writes

$$\widetilde{C}^l_t = \Big\{(\gamma,\beta) \in L^2(\lambda) \times \mathbb{R}^n : \gamma \geq -\zeta_t^{1/2}, \ \big\|2g^l(1+\zeta_t^{-\frac{1}{2}}\gamma)\big\|_{L^1(\lambda_t)} + |\beta|^2 \leq K_t^2\Big\}.$$

For any $l \in \mathbb{N}$ since $g^l \leq g$, then $C_t \subseteq \bar{C}^l_t$, which implies $\bar{C}_t \subseteq \bar{C}^l_t$. Hence $\widetilde{C}_t \subseteq \bigcap_{l\in\mathbb{N}} \widetilde{C}^l_t$, $t \in [0,T]$. In fact equality holds in the latter inclusion as claimed by the following lemma, which also infers that $\widetilde{C}$ satisfies Assumption 2.9. The proof is provided in Appendix 2.5.

Lemma 2.30. *For C defined in (2.57), holds $\widetilde{C}_t = \bigcap_{l\in\mathbb{N}} \widetilde{C}^l_t, \ t \in [0,T]$. In particular the closed-valued correspondence $\widetilde{C}$ is $\overline{\mathcal{P}}$-measurable.*

Overall, the correspondence C defined in (2.57) and describing a no-good-deal constraint as a bound on the optimal expected growth rates in the financial market satisfies the hypotheses of Theorem 2.26. This yields an approximation of the associated good-deal bound $\pi^u_\cdot(X)$ in terms of solutions to Lipschitz JBSDEs for abstract random measures μ and contingent claims $X \in L^\infty$. Although the correspondence C in (2.57) might not satisfy Assumption 2.13 for general random measures μ, this assumption apparently holds when the measures λ_t (with $\nu(dt,de) = \lambda_t(de)dt$ respect to which the compensator ν is absolutely continuous) are finitely supported. In this case the results of Section 2.3 (on valuation and hedging) are again applicable, and the good-deal bounds can be directly described as solutions to JBSDEs. Without loss of generality, we elaborate on this by considering the semi-Markov setup of Section 2.3.1.

Example for semi-Markov jump-dynamics: Consider again the framework of Section 2.3.1, with a semi-Markov process J on finite state space $E = \{e_1, \ldots, e_m\} \subset \mathbb{R}^m$, and denote $\mathcal{I} = \{1,\ldots,m\}$, and counting process $N = \sum_{i,j\in\mathcal{I}, j\neq i} N^{ij}$ of the jumps, compensator $\nu(dt) := \sum_{i,j\in\mathcal{I}, j\neq i} \lambda^{ij}_t dt$, for jump intensities $\lambda^{ij}_t = \mathbb{1}_{\{J_{t-}=e_i\}}\alpha^{ij}_t(\tau_{t-})$ with $\alpha^{ij} \geq 0$ defined in (2.33) and the time the process has spent at state J_t being $\tau_t := \sup\{s \in [0,t] : J_{t-u} = J_t, \ u \in [t-s,t]\}$. The constraint in the optimization problem in Lemma 2.14 for C in (2.57) is finite dimensional and given for $t \in [0,T]$ by the set of $(\gamma,\eta) \in (-1,\infty)^{m\times m-1} \times \mathbb{R}^n$ satisfying

$$\frac{1}{2}|\eta|^2 + \sum_{i,j\in\mathcal{I}, j\neq i} g(1+\gamma^{ij})\lambda^{ij}_t \leq \frac{1}{2}(K_t^2 - |\xi_t|^2), \tag{2.58}$$

for g given by (2.3). We can assume without loss of generality $\gamma^{ij} = 0$ on the set $\{\lambda^{ij} = 0\}$. This together with (2.58) imply that

$$g(1+\gamma^{ij}) \leq \frac{K_t^2 - |\xi_t|^2}{2\lambda_t^{ij}} \mathbb{1}_{\{\lambda_t^{ij} \neq 0\}} \quad \text{for all } i, j \in \mathcal{I},\, j \neq i. \tag{2.59}$$

Since g is continuous and $\lim_{x \searrow -1} g(1+x) = \lim_{x \to \infty} g(1+x) = +\infty$, then (2.59) yields compactness in $(-1,\infty)^{m \times m-1} \times \mathbb{R}^n$ of the set of values of $((\gamma^{ij})_{i,j \in \mathcal{I},\, j \neq i}, \eta)$ satisfying (2.58). Furthermore for $Z \in \mathcal{H}^2$ and $U \in \mathcal{H}^2_\nu$, the objective function $F(t,\gamma,\eta) := \eta^{\text{tr}} \Pi_t^{\perp}(Z_t) + \sum_{i,j \in \mathcal{I},\, j \neq i} U_t^{ij} \gamma^{ij} \lambda_t^{ij}$ is continuous in $(\gamma,\eta) \in (-1,\infty)^{m \times m-1} \times \mathbb{R}^n$ and predictable in $(t,\omega) \in [0,T] \times \Omega$. Hence by the usual direct method of variational analysis (cf. [ET99]) and standard measurable selection theorems (cf. [Roc76], which do not require completeness of the measure space for correspondences with finite dimensional ranges), there exists a predictable (t,ω)-wise maximizer $(\bar{\gamma},\bar{\eta}) := (\bar{\gamma}(Z,U), \bar{\eta}(Z,U)) \in (-1,\infty)^{m \times m-1} \times \mathbb{R}^n$ of F over the constraint set described by (2.58). Since by part 3. of Lemma 2.32 the function g satisfies

$$|x| - 2 \leq (g(1+x))^2, \text{ for all } x > -1,$$

then (2.59) implies that $|\gamma^{ij}| \leq \frac{(K_t^2 - |\xi_t|^2)^2}{4(\lambda_t^{ij})^2} \mathbb{1}_{\{\lambda_t^{ij} \neq 0\}} + 2$. This in turn yields (after squaring and summing over all states $e_i, e_j \in E$, $j \neq i$)

$$\begin{aligned} \sum_{i,j \in \mathcal{I},\, j \neq i} \left|\gamma^{ij}\right|^2 \lambda_t^{ij} &\leq \sum_{i,j \in \mathcal{I},\, j \neq i} \Big(\frac{(K_t^2 - |\xi_t|^2)^2}{4(\lambda_t^{ij})^2} \mathbb{1}_{\{\lambda_t^{ij} \neq 0\}} + 2 \Big)^2 \lambda_t^{ij} \\ &\leq \Big(\frac{|K|_\infty^4}{4} \vee 2 \Big) \sum_{i,j \in \mathcal{I},\, j \neq i} \Big((\lambda_t^{ij})^{-2} \mathbb{1}_{\{\lambda_t^{ij} \neq 0\}} + 1 \Big)^2 \lambda_t^{ij}, \end{aligned} \tag{2.60}$$

with the convention that $0/0 = 0$. Now assume that

$$\exists\, \bar{c}_\lambda \geq \underline{c}_\lambda > 0 \text{ s.t. } \underline{c}_\lambda \mathbb{1}_{\{\lambda^{ij} \neq 0\}} \leq \lambda^{ij} \leq \bar{c}_\lambda \quad \text{for all } i, j \in \mathcal{I},\, j \neq i. \tag{2.61}$$

Condition (2.61) ensures by (2.60) that the correspondence C defined in (2.57) satisfies the uniform boundedness Assumption 2.13 in the current semi-Markov jump setup with

$$|\beta|^2 + \sum_{i,j \in \mathcal{I},\, j \neq i} |\gamma^{ij}|^2 \lambda_t^{ij} \leq |K|_\infty^2 + m(m-1) \Big(\frac{|K|_\infty^4}{4} \vee 2 \Big) \Big(\frac{1}{\underline{c}_\lambda^2} + 1 \Big)^2 \bar{c}_\lambda, \tag{2.62}$$

for all $(\gamma,\beta) \in C$. Note that (2.61) does not exclude the fact that the intensities λ^{ij} can vanish on a non-negligible set. Indeed we only require on the set where they do not vanish, that they are bounded from below by a positive constant, uniformly over all states $e_i, e_j \in E$ and $(t,\omega) \in [0,T] \times \Omega$. That the intensities of the jumps are bounded from above appears as a non-restrictive assumption for practical examples, which could be interpreted as a sufficient condition preventing the rate of state-change of the semi-Markov process from exploding.

Former calculations suggest that in the limit as $m \to \infty$, the right-hand side of in (2.62) would tend to infinity. In the limiting case, therefore, the correspondence C may no longer be uniformly bounded; this shows the importance of restricting to finitely many states.

Part a) of Theorem 2.16 applies and yields that the good-deal bound $\pi^u_\cdot(X)$ for $X \in L^2$ is given by $\pi^u_\cdot(X) = E^{\bar{Q}}_\cdot[X] = Y$ for $\bar{Q} = Q^{\bar{\gamma}, -\xi+\bar{\eta}}$ and $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ solution to the BSDE (2.18) which in the present setup rewrites

$$\begin{aligned} Y_t = X + \int_t^T \Big((-\xi_s + \bar{\eta}_s)^{\mathrm{tr}} Z_s + \sum_{i,j\in\mathcal{I},\, j\neq i} U^{ij}_s \bar{\gamma}^{ij}_s \lambda^{ij}_s \Big) ds \\ - \int_t^T Z^{\mathrm{tr}}_s dW_s - \sum_{i,j\in\mathcal{I},\, j\neq i} \int_t^T U^{ij}_s d\widetilde{N}^{ij}_s. \end{aligned} \tag{2.63}$$

In addition, the worst-case measure $\bar{Q} = Q^{\bar{\gamma}, -\xi+\bar{\eta}}$ is in fact a no-good-deal measure, i.e. $\bar{Q} \in \mathcal{Q}^{\mathsf{ngd}}$, because the optimal Girsanov kernels $(\bar{\gamma}, -\xi + \bar{\eta}) \in C$ satisfies $\bar{\gamma}^{ij} > -1$ for all $i, j \in \mathcal{I}$, $j \neq i$. It is also possible to obtain a qualitative result about good-deal hedging in this setting. Indeed, condition (2.22) is clearly satisfied for $\epsilon = \varepsilon \in (0, 1)$ since by assumption $K > |\xi| + \varepsilon$. Hence applying Theorem 2.19 yields in particular existence of a good-deal hedging strategy $\bar{\phi} = \bar{\phi}(X)$ (for $X \in L^2$) with

$$f^{\bar{\phi}}(t, Z_t, U_t) = \operatorname*{ess\,inf}_{\phi\in\Phi} f^{\phi}(t, Z_t, U_t), \tag{2.64}$$

for (Y, Z, U) solution to the BSDE (2.63) and

$$f^{\phi}(t, Z_t, U_t) = -\xi^{\mathsf{tr}}_t \phi_t + \operatorname*{ess\,sup}_{(\gamma,\beta)} \Big(\beta^{\mathsf{tr}} (Z_t - \phi_t) + \sum_{i,j\in\mathcal{I},\, j\neq i} U^{ij}_t \gamma^{ij} \lambda^{ij}_t \Big),$$

where the supremum is taken over all $(\gamma, \beta) = ((\gamma^{ij})_{i,j\in\mathcal{I},\, j\neq i}, \beta) \in (-1, \infty)^{m\times m-1} \times \mathbb{R}^n$ satisfying (2.58). One could not expect to obtain for (2.64), in the generality of the present example, an explicit formula for the good-deal hedging strategy $\bar{\phi}$ solving the minimization problem in (2.64). Yet, approximations might be computed using numerical algorithms for convex optimization problems (cf. e.g. [BV04]), and of Lipschitz JBSDEs (cf. e.g. [BE08] for related but different types of generators).

2.5 Appendix

This appendix collects some proofs and statements of results that were omitted in the main body of the chapter. The order of appearance here is the same as in the main text.

Proof of Lemma 2.1. For part a), apply [JS03, Theorem III.3.24] to $\tilde{X}$ with P-characteristics $(B, c, \nu) := (0, I, \nu^P)$ with respect to the truncation function h. Note that $\tilde{X}$ has a canonical representation $\tilde{X} = W + (\mathrm{Id}_E - h) * \mu + h * \widetilde{\mu}$ in terms of the truncation function h, where Id_E is the identity function on E. Part b) is a consequence of part a), the weak predictable representation property (2.2) of $(W, \widetilde{\mu})$ with respect to $(P, \mathbb{F})$, and [JS03, Proposition III.5.10, Theorem III.5.19 and Corollary III.5.22] which apply with $Y := 1 + \gamma$, and $a = 0$, $\hat{Y} = 0$ since $\nu \ll \lambda \otimes dt$. □

Proposition 2.31 ([LM78], Theorem II.5). *Let M be a quasi-left-continuous local martingale satisfying $\Delta M \geq -1$ and define $\bar{T} := \inf\{t : \Delta M_t = -1\} \wedge T$. If the predictable compensator Λ of the process*

$$D = \langle M^c \rangle_{\cdot \wedge \bar{T}} + \sum_{s \leq \cdot \wedge \bar{T}} \Delta M_s^2 \mathbb{1}_{\{|\Delta M_s| \leq 1\}} + \Delta M_s \mathbb{1}_{\{\Delta M_s > 1\}} \tag{2.65}$$

is bounded, then $E\left[\left([\mathcal{E}(M)]_T\right)^{1/2}\right] < \infty$. In particular $\mathcal{E}(M)$ is a uniformly integrable martingale.

We have the following lemma. Being purely analytical, the proof is omitted.

Lemma 2.32. *For any $y \geq 0$, hold*

1. $(1 - \sqrt{y})^2 \leq (y-1)^2 \mathbb{1}_{\{y \leq 2\}} + |y - 1| \mathbb{1}_{\{y > 2\}} \leq \dfrac{1}{(1-\sqrt{2})^2}(1 - \sqrt{y})^2$,
2. $(1 - \sqrt{y})^2 \leq g(y)$, *for the function g defined in (2.3),*
3. $|y - 1| - 2 \leq (g(y))^2$.

Proof of Proposition 2.3. By Lemma 2.32 it follows that

$$0 \leq \int_E (1 - \sqrt{1 + \gamma_t(e)})^2 \lambda_t(de) \leq \int_E g(1 + \gamma_t(e)) \lambda_t(de) \leq K, \ t \in [0, T].$$

Hence the process $\left(1 - \sqrt{1+\gamma}\right)^2 * \nu$ is locally P-integrable. Then applying [JS03, Theorem II.1.33, d)] (with $a = 0$ and $\widehat{\gamma} = 0$ since $\nu \ll \lambda \otimes dt$ holds) yields the $\widetilde{\mu}$-integrability of γ, with $\gamma * \widetilde{\mu}$ being a purely discontinuous local martingale. Moreover by (2.5), $\beta \cdot W$ is well-defined as a continuous local martingale, so that M is a local martingale with $M^c = \beta \cdot W$ and $M^d = \gamma * \widetilde{\mu}$. By definition, the jumps of M are given by $\Delta M_t = \Delta\left(\gamma * \widetilde{\mu}\right)_t = \gamma(t, \Delta \widetilde{X}_t) \mathbb{1}_{\{\Delta \widetilde{X}_t \neq 0\}}$, $t \in [0, T]$, where $\widetilde{X}$ is the semimartingale $\widetilde{X} = W + (\mathrm{Id}_E - h) * \mu + h * \widetilde{\mu}$ with $h(e) := e \mathbb{1}_{\{|e| \leq 1\}}$. Now since $\gamma > -1$ then $\Delta M > -1$, and therefore $\mathcal{E}(M)$ is a positive local martingale. By [JS03, Corollary II.1.19] and $\nu \ll \lambda \otimes dt$, M is quasi-left-continuous. Hence the process D in (2.65) can be written as $D = \int_0^\cdot |\beta_s|^2 ds + \left(\gamma^2 \mathbb{1}_{\{|\gamma| \leq 1\}} + \gamma \mathbb{1}_{\{\gamma > 1\}}\right) * \mu$. By [JS03, Proposition II.1.28] the predictable P-compensator Λ of D is $\Lambda = \int_0^\cdot |\beta_s|^2 ds + \left(\gamma^2 \mathbb{1}_{\{|\gamma| \leq 1\}} + \gamma \mathbb{1}_{\{\gamma > 1\}}\right) * \nu$. Now using (2.4) and (2.5), Lemma 2.32 yields boundedness of Λ. By Proposition 2.31 this implies that $\Gamma = \mathcal{E}(M)$ is a positive uniformly integrable martingale. In particular Γ defines a measure $Q \sim P$ via $dQ = \Gamma dP$. Let (β^Q, γ^Q) be the actual Girsanov kernels of Q from part a) of Lemma 2.1. By part b) of Lemma 2.1, $\Gamma = \mathcal{E}(M^Q)$ holds with $M^Q = \beta^Q \cdot W + \gamma^Q * \widetilde{\mu}$. Hence $\mathcal{E}(M) = \mathcal{E}(M^Q)$ and by taking stochastic logarithms one obtains $M = M^Q$, or equivalently $(\beta^Q - \beta) \cdot W = (\gamma^Q - \gamma) * \widetilde{\mu}$. The left hand side is a continuous local martingale whereas the right hand side is a purely discontinuous local martingale. By orthogonality both local martingales are equal to zero. Since from (2.5) both local martingales are square integrable, then $\beta = \beta^Q$ $P \otimes dt$-a.s. and $\gamma = \gamma^Q$ $P \otimes \lambda \otimes dt$-a.s.. $\square$

Proof of Lemma 2.11. For $(\gamma^i, \beta^i) \in C$ and $\beta^i = -\xi + \eta^i$, $\eta^i \in \operatorname{Ker}\sigma$, $i = 1, 2$, let Q^{γ^i,β^i} be in $\mathcal{Q}^{\mathsf{ngd}}$ with density processes Γ^i with respect to P given by $\Gamma^i := \mathcal{E}\left((-\xi + \eta^i) \cdot W + \gamma^i * \widetilde{\mu}\right)$.

Convexity: Let $\alpha \in [0,1]$ and $\Gamma = \alpha\Gamma^1 + (1-\alpha)\Gamma^2$. Since $\mathcal{M}^e$ is convex, then $\Gamma \in \mathcal{M}^e$ and corresponds to a measure $Q^{\gamma,\beta} \sim P$ with Girsanov kernels $(\gamma, \beta = -\xi + \eta)$, where $\eta \in \operatorname{Ker}\sigma$. Using Itô's formula and convexity of the values of C one shows that $(\gamma_t, \beta_t) = \frac{\alpha\Gamma^1_{t-}}{\Gamma_{t-}}(\gamma^1_t, \beta^1_t) + \frac{(1-\alpha)\Gamma^2_{t-}}{\Gamma_{t-}}(\gamma^2_t, \beta^2_t) \in C_t,\ t \in [0,T]$. Hence $\mathcal{Q}^{\mathsf{ngd}}$ is convex.

M-stability: Let $\tau \leq T$ be a stopping time and $\Gamma_t := \mathbb{1}_{\{t \leq \tau\}}\Gamma^1_t + \mathbb{1}_{\{\tau \leq t\}}\Gamma^1_\tau\Gamma^2_t/\Gamma^2_\tau$, $t \in [0,T]$. Since $\mathcal{M}^e$ is m-stable, then $\Gamma \in \mathcal{M}^e$ and corresponds to a measure $Q^{\gamma,\beta} \sim P$ with Girsanov kernels $(\gamma, \beta := -\xi + \eta)$, where $\eta \in \operatorname{Ker}\sigma$. We show that $(\gamma, \beta) \in C$. It holds that $\log\Gamma^i_t = \beta^i \cdot W_t + (\log(1+\gamma^i)) * \widetilde{\mu}_t - \int_0^t \left(\frac{1}{2}|\beta^i_s|^2 + \int_E g(1+\gamma^i_s(e))\lambda_s(de)\right)ds$, for $i = 1, 2$, and g being the function given by (2.3). Hence

$$\begin{aligned}\log\Gamma_T = \beta^2 \cdot W_T + (\log(1+\gamma^2)) * \widetilde{\mu}_T - \int_0^T \Big(\frac{1}{2}|\beta^2_s|^2 + \int_E g(1+\gamma^2_s(e))\lambda_s(de)\Big)ds \\ + (\beta^1 - \beta^2) \cdot W_\tau + (\log(1+\gamma^1) - \log(1+\gamma^2)) * \widetilde{\mu}_\tau \\ - \int_0^\tau \Big(\frac{1}{2}(|\beta^1_s|^2 - |\beta^2_s|^2) + \int_E (g(1+\gamma^1_s(e)) - g(1+\gamma^2_s(e))\lambda_s(de)\Big)ds.\end{aligned}$$

Equivalently we have

$$\begin{aligned}&\beta \cdot W_T + (\log(1+\gamma) * \widetilde{\mu}_T - \int_0^T \Big(\frac{1}{2}|\beta_s|^2 + \int_E g(1+\gamma_s(e))\lambda_s(de)\Big)ds \\ &= \beta^1 \cdot W_\tau + (\log(1+\gamma^1)) * \widetilde{\mu}_\tau - \int_0^\tau \Big(\frac{1}{2}|\beta^1_s|^2 + \int_E g(1+\gamma^1_s(e))\lambda_s(de)\Big)ds \\ &\qquad + \int_\tau^T \beta^2_s dW_s + \int_\tau^T \int_E (\log(1+\gamma^2_s(e))\widetilde{\mu}(ds, de) \\ &\qquad - \int_\tau^T \Big(\frac{1}{2}|\beta^2_s|^2 + \int_E g(1+\gamma^2_s(e))\lambda_s(de)\Big)ds \\ &= \Big(\mathbb{1}_B\beta^1 + \mathbb{1}_{B^c}\beta^2\Big) \cdot W_T + \Big(\log\Big(1 + \mathbb{1}_B(s)\gamma^1 + \mathbb{1}_{B^c}(s)\gamma^2\Big)\Big) * \widetilde{\mu}_T \\ &\quad - \int_0^T \Big(\frac{1}{2}|\mathbb{1}_B(s)\beta^1_s + \mathbb{1}_{B^c}(s)\beta^2_s|^2 + \int_E g\Big(1 + \mathbb{1}_B(s)\gamma^1_s(e) + \mathbb{1}_{B^c}(s)\gamma^2_s(e)\Big)\lambda_s(de)\Big)ds,\end{aligned}$$

where $B = [0, \tau] = \{(t, \omega) : \ t \leq \tau(\omega)\} \in \mathcal{P}$. Thus $(\gamma, \beta) = \mathbb{1}_B(\gamma^1, \beta^1) + \mathbb{1}_{B^c}(\gamma^2, \beta^2) \in C$ since C has convex values. $\square$

Proof of Lemma 2.14. Proofs for part a) and b) are analogous, so we only prove part a). Consider the equivalent (to (2.17)) maximization problem

$$(\gamma_t^*(\omega), \eta_t^*(\omega)) = \underset{(\gamma_t(\omega), \eta_t(\omega))}{\operatorname{argmax}} \ \eta_t^{\mathrm{tr}}(\omega) \Pi_{(t,\omega)}^{\perp}(Z_t(\omega)) + \int_E U_t(\omega, e)\gamma_t(\omega, e)(\zeta_t(\omega, e))^{1/2} \lambda(de), \quad (2.66)$$

where the maximum is taken over $(\gamma_t(\omega), \eta_t(\omega)) \in \widetilde{C}_t(\omega) + (0, \xi_t(\omega))$ and $\eta_t(\omega) \in \operatorname{Ker} \sigma_t(\omega)$, with $\widetilde{C}$ given in (2.11). The maximization problem (2.66) is more convenient for measurable selection arguments since the range $L^2(\lambda) \times \mathbb{R}^n$ of the associated correspondence $\widetilde{C}$ does not depend on t nor ω. The corresponding maximizers of (2.17) and (2.66) are related by

$$(\bar{\gamma}_t, \bar{\eta}_t) = \Big(\frac{\gamma_t^*}{\sqrt{\zeta_t}} \mathbb{1}_{\{\zeta_t > 0\}}, \eta_t^*\Big), \ t \in [0, T]. \quad (2.67)$$

For all $(t, \omega) \in [0, T] \times \Omega$, the sets $\widetilde{C}_t(\omega)$ are closed and convex in $L^2(\lambda) \times \mathbb{R}^n$ since $\bar{C}_t(\omega)$ are closed and convex in $L^2(\lambda_t(\omega)) \times \mathbb{R}^n$. In addition $\widetilde{C}_t(\omega)$ are bounded in $L^2(\lambda) \times \mathbb{R}^n$ (hence weakly compact) since by Assumption 2.13 the sets $\bar{C}_t(\omega)$ are bounded in $L^2(\lambda_t(\omega)) \times \mathbb{R}^n$. As a consequence $\big(\widetilde{C}_t(\omega) + (0, \xi_t(\omega))\big) \cap \big(L^2(\lambda) \times \operatorname{Ker} \sigma_t(\omega)\big)$ is also weakly compact in $L^2(\lambda) \times \mathbb{R}^n$. Since $Z \in \mathcal{H}^2$ and $U \in \mathcal{H}^2_\nu$, the objective function of the maximization problem (2.66) is linear and continuous in $(\gamma, \eta) \in L^2(\lambda) \times \mathbb{R}^n, \ t \in [0, T]$. Hence by the direct method in variational analysis (see [ET99]), there exists for all $(t, \omega) \in [0, T] \times \Omega$ a maximizer $(\gamma_t^*(\omega), \eta_t^*(\omega))$ in $\big(\widetilde{C}_t(\omega) + (0, \xi_t(\omega))\big) \cap \big(L^2(\lambda) \times \operatorname{Ker} \sigma_t(\omega)\big)$. Now we show that one can choose (γ^*, η^*) such that η^* is $\overline{\mathcal{P}}$-measurable and γ^* (hence $\bar{\gamma} = \gamma^* \zeta^{-1/2} \mathbb{1}_{\{\zeta > 0\}}$) is $\overline{\mathcal{P}} \otimes \mathcal{E}$-measurable (since ζ clearly is). Note that the Hilbert space $L^2(\lambda)$ is separable (hence is a Polish space) and also that the correspondence $\big(\widetilde{C} + (0, \xi)\big) \cap \big(L^2(\lambda) \times \operatorname{Ker} \sigma\big)$ is $\overline{\mathcal{P}}$-measurable since $\widetilde{C}$ is $\overline{\mathcal{P}}$-measurable by Assumption 2.9 and ξ, σ are predictable processes. Since Z is predictable and U is $\widetilde{\mathcal{P}}$-measurable, the objective function is $\overline{\mathcal{P}}$-measurable in $(t, \omega) \in [0, T] \times \Omega$ and hence a Carathéodory function defined on $[0, T] \times \Omega \times L^2(\lambda) \times \mathbb{R}^n$. By standard measurable selection [AF90, Theorems 8.1.3, 8.2.11] one obtains (γ^*, η^*) satisfying (2.66) for all $(t, \omega) \in [0, T] \times \Omega$, with η^* $\overline{\mathcal{P}}$-measurable and γ^* $\overline{\mathcal{P}} - \mathcal{B}(L^2(\lambda))$-measurable. Let us show that γ^* defined by $\gamma^*(t, \omega, e) := \gamma^*(t, \omega)(e)$ is actually $\overline{\mathcal{P}} \otimes \mathcal{E}$-measurable. Denote by $(u^n)_{n \in \mathbb{N}}$ an orthonormal basis of $L^2(\lambda)$. Then $\gamma_t^*(\omega)$ has the decomposition $\gamma_t^*(\omega) = \sum_{n \in \mathbb{N}} \langle \gamma_t^*(\omega), u^n \rangle_{L^2(\lambda)} u^n$ for any $(t, \omega) \in [0, T] \times \Omega$. Now since for each $n \in \mathbb{N}$ the map $L^2(\lambda) \ni \gamma \mapsto \langle \gamma, u^n \rangle_{L^2(\lambda)}$ is continuous, then $\langle \gamma^*, u^n \rangle_{L^2(\lambda)}$ is a $\overline{\mathcal{P}}$-measurable process for all $n \in \mathbb{N}$. Thus γ^* is $\overline{\mathcal{P}} \otimes \mathcal{E}$-measurable as a countable sum of $\overline{\mathcal{P}} \otimes \mathcal{E}$-measurable functions. Now by approximation of measurable functions by simple functions, one can make η^* predictable and γ^* $\widetilde{\mathcal{P}}$-measurable through modification on a $P \otimes dt$-null-set. The corresponding $(\bar{\gamma}, \bar{\eta})$ given by (2.67) then solves (2.17) for $P \otimes dt$-almost all $(t, \omega) \in [0, T] \times \Omega$. $\square$

Proof of Theorem 2.16. Part a) and b) are analogous, so we only prove part a). Denote by f the generator of the JBSDE (2.18). For all $t \in [0,T]$, f_t is, by part a) of Lemma 2.14, the supremum over $(\gamma, -\xi+\eta) \in \bar{C}$ and $\eta \in \operatorname{Ker}\sigma$ of a family of linear generators

$$f_t^{\gamma,\eta}(t,z,u) := (-\xi_t + \eta_t)^{\mathrm{tr}} z + \int_E u(e)\gamma_t(e)\lambda_t(de),$$

where coefficients $(\gamma, (-\xi+\eta))$ are bounded in $\mathbb{R}^n \times L^2(\lambda_t(\omega))$ uniformly in (t,ω) by $K_f := \sup_{(t,\omega)} \sup_{(\gamma,\beta)\in\bar{C}(t,\omega)} \left(\|\gamma\|_{L^2(\lambda_t)} + |\beta|\right) \in (0,\infty)$. The generator f is then Lipschitz continuous in $(z,u) \in \mathbb{R}^n \times L^2(\lambda_t(\omega))$, uniformly in $(t,\omega) \in [0,T]\times\Omega$ with Lipschitz constant K_f and satisfies $f_t(0,0) = 0$. By [Bec06, Proposition 3.2], the JBSDE (2.18) has a unique solution $(Y,Z,U) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$. Now recall that for each $(\gamma,\beta) \in \bar{C}$, the process β is bounded and $\int_E \gamma_t^2(e)\lambda_t(de)$ is uniformly bounded in $t \in [0,T]$. Hence by Lemma 2.4 and the subsequent remark, the JBSDEs with generators $f^{\gamma,\eta}$ have unique solutions $(Y^{\gamma,\eta}, Z^{\gamma,\eta}, U^{\gamma,\eta}) \in \mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2_\nu$, which satisfy $Y_t^{\gamma,\eta} = E_t^{Q^{\gamma,\beta}}[X]$, $t\in[0,T]$, for $\beta = -\xi+\eta$. Furthermore since $f = f^{\bar{\gamma},\bar{\eta}}$ holds, then one also has $Y_t = E_t^{\bar{Q}}[X]$, $\bar{Q}$-a.s.. From Lemma 2.15 it holds that $\pi_t^u(X) \geq E_t^{\bar{Q}}[X]$, $\bar{Q}$-a.s., and so to conclude the proof, one has to show that $\pi_t^u(X) \leq Y_t$, P-a.s.. For all $(\gamma, \beta := -\xi+\eta) \in C$ (defining $Q^{\gamma,\beta} \in \mathcal{Q}^{\mathsf{ngd}}$) holds

$$f_t(Z_t,U_t) = f_t^{\bar{\gamma},\bar{\eta}}(Z_t,U_t) \geq f_t^{\gamma,\eta}(Z_t,U_t), \quad t\in[0,T],$$

for (Y,Z,U) solution to the JBSDE (2.18). Moreover since $f^{\gamma,\eta}$ are Lipschitz in (z,u) with uniform Lipschitz constants K_f and

$$f_t^{\gamma,\eta}(Z_t^{\gamma,\eta}, U_t^{\gamma,\eta}) - f_t^{\gamma,\eta}(Z_t^{\gamma,\eta}, U_t) = \int_E \gamma_t(e)(U_t^{\gamma,\eta}(e) - U_t(e))\lambda_t(de), \quad t\in[0,T],$$

with $\mathcal{E}\left((-\xi+\eta)\cdot W + \gamma * \widetilde{\mu}\right)$ being a uniformly integrable martingale by Proposition 2.31, then Proposition 2.6 implies that $Y_t \geq Y_t^{\gamma,\eta}$, P-a.s., for all $(\gamma,\beta = -\xi+\eta) \in C$. As a consequence $Y_t \geq \text{ess sup}_{(\gamma,\eta)} Y_t^{\gamma,\eta} = \pi_t^u(X)$, P-a.s., where $(\gamma, -\xi+\eta)$ range over all Girsanov kernels of measures $Q \in \mathcal{Q}^{\mathsf{ngd}}$. □

Proof of Lemma 2.17. Denote by f^ϕ (for $\phi\in\Phi$) the generator of the JBSDE (2.20). By part b) of Lemma 2.14, the generator f^ϕ is the supremum of a family of affine JBSDE generators $f_t^{(\phi,\gamma,\beta)}(t,z,u) = -\xi_t^{\mathrm{tr}}\phi_t + (Z_t-\phi_t)^{\mathrm{tr}}\beta_t + \int_E U_t(e)\gamma_t(e)\lambda_t(de)$, with coefficients $(\gamma,\beta) \in \bar{C}$ bounded in $\mathbb{R}^n \times L^2(\lambda_t(\omega))$ uniformly in $(t,\omega)\in[0,T]\times\Omega$ by the constant $K_f := \sup_{(t,\omega)}\sup_{(\gamma,\beta)\in\bar{C}(t,\omega)}\left(\|\gamma\|_{L^2(\lambda_t)} + |\beta|\right) \in (0,\infty)$. Hence for all $\phi\in\Phi$, f^ϕ is Lipschitz continuous in $(z,u)\in\mathbb{R}^n\times L^2(\lambda_t(\omega))$ uniformly in $(t,\omega)\in[0,T]\times\Omega$ with Lipschitz constant K_f, and satisfies $f_t^\phi(0,0)\in\mathcal{H}^2$ since ξ is bounded and $\phi\in\mathcal{H}^2$. By [Bec06, Proposition 3.2], the JBSDE (2.20) has a unique solution $(Y^\phi, Z^\phi, U^\phi) \in \mathcal{S}^2\times\mathcal{H}^2\times\mathcal{H}^2_\nu$.

Now let $\widetilde{Y} := Y^{\phi} - \phi \cdot \widehat{W}$ and $\widetilde{Z} = Z - \phi$, so that using (2.20) gives

$$\begin{aligned}-d\widetilde{Y}_t =& - dY_t^{\phi} + \xi_t^{\mathrm{tr}}\phi_t dt + \phi_t^{\mathrm{tr}} dW_t \\ =&\Big(\widetilde{Z}_t^{\mathrm{tr}}\tilde{\beta}_t(\widetilde{Z},U) + \int_E U_t(e)\tilde{\gamma}_t(\widetilde{Z},U)(e)\lambda_t(de)\Big)dt - \widetilde{Z}_t^{\mathrm{tr}} dW_t - \int_E U_t(e)\widetilde{\mu}(dt,de).\end{aligned}$$

This means that $(\widetilde{Y},\widetilde{Z})$ solves the JBSDE (2.19) with terminal value $\widetilde{Y}_T = X - \phi \cdot \widehat{W}_T \in L^2$. Hence part b) of Theorem 2.16 implies that $\widetilde{Y}_t = \rho_t\Big(X - \phi \cdot \widehat{W}_T\Big)$. Finally translation invariance of ρ yields $Y_t^{\phi} = \widetilde{Y}_t + \phi \cdot \widehat{W}_t = \rho_t\left(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\right),\ t \in [0,T]$. □

Proof of Lemma 2.18. Consider deterministic (and time-independent) parameters $z \in \mathbb{R}^n$, $u \in L^2(\zeta\lambda)$, $\sigma \in \mathbb{R}^{d\times n}$, $\xi \in \operatorname{Im}\sigma^{\mathsf{tr}}$, and for a convex closed and bounded set $\bar{C} \subseteq L^2(\zeta\lambda) \times \mathbb{R}^n$, consider the function $L : \mathbb{R}^n \times (L^2(\zeta\lambda) \times \mathbb{R}^n) \to \mathbb{R}$ defined by

$$(\phi,(\gamma,\beta)) \mapsto L(\phi,(\gamma,\beta)) := -\xi^{\mathrm{tr}}\phi + \beta^{\mathrm{tr}}(z-\phi) + \int_E u(e)\gamma(e)\zeta(e)\lambda(de).$$

Clearly for any fixed $\phi \in \mathbb{R}^n$ the function $(\gamma,\beta) \mapsto L(\phi,(\gamma,\beta))$ is linear and bounded, and for any fixed (γ,β) the function $\phi \mapsto L(\phi,(\gamma,\beta))$ is linear and continuous. Since the set $\bar{C}$ is convex closed and bounded, it is weakly compact in $L^2(\zeta\lambda) \times \mathbb{R}^n$. Now since $\operatorname{Im}\sigma^{\mathsf{tr}}$ is convex and closed, then by [ET99, Proposition 2.3, Chapter VI] the minimax identity

$$\inf_{\phi \in \operatorname{Im}\sigma^{\mathsf{tr}}} \sup_{(\gamma,\beta)\in\bar{C}} L(\phi,(\gamma,\beta)) = \sup_{(\gamma,\beta)\in\bar{C}} \inf_{\phi \in \operatorname{Im}\sigma^{\mathsf{tr}}} L(\phi,(\gamma,\beta)) \tag{2.68}$$

holds. Plus, the right hand side of (2.68) is equal to

$$\begin{aligned}\sup_{(\gamma,\beta)\in\bar{C}} \inf_{\phi \in \operatorname{Im}\sigma^{\mathsf{tr}}} L(\phi,(\gamma,\beta)) &= \sup_{(\gamma,\beta)\in\bar{C}} \Big(\beta^{\mathrm{tr}} z + \int_E u(e)\gamma(e)\zeta(e)\lambda(de) + \inf_{\phi \in \operatorname{Im}\sigma^{\mathsf{tr}}} \phi^{\mathsf{tr}}(\xi + \Pi(\beta))\Big) \\ &= \sup_{\substack{(\gamma,\beta)\in\bar{C} \\ \Pi(\beta)=-\xi}} \beta^{\mathrm{tr}} z + \int_E u(e)\gamma(e)\zeta(e)\lambda(de),\end{aligned}$$

since $\inf_{\phi \in \operatorname{Im}\sigma^{\mathsf{tr}}} \phi^{\mathsf{tr}}(\xi + \Pi(\beta))$ equals 0 if $\Pi(\beta) = -\xi$ and $-\infty$ otherwise. Now extending the arguments to random and time-dependent parameters clearly gives (2.21). □

Proof of Theorem 2.19. Consider deterministic (and time-independent) parameters $z \in \mathbb{R}^n$, $u \in L^2(\zeta\lambda)$, $\sigma \in \mathbb{R}^{d\times n}$, $\xi \in \operatorname{Im}\sigma^{\mathsf{tr}}$, and for a convex bounded set $\bar{C} \subseteq L^2(\zeta\lambda) \times \mathbb{R}^n$ satisfying $\{0\} \times B_\epsilon(-\xi) \subseteq \bar{C}$, consider the following analog of f^{ϕ} as a function of ϕ:

$$\mathbb{R}^n \supseteq \operatorname{Im}\sigma^{\mathsf{tr}} \ni \phi \mapsto F(\phi) := -\xi^{\mathrm{tr}}\phi + \operatorname*{ess\,sup}_{(\gamma,\beta)\in\bar{C}} \big(\beta^{\mathrm{tr}}(z-\phi) + \int_E u(e)\gamma(e)\zeta(e)\lambda(de)\big).$$

The function F is clearly convex and continuous. Moreover F is coercive on $\operatorname{Im}\sigma^{\mathsf{tr}}$, i.e. $F(\phi) \to \infty$ as $|\phi| \to \infty$ for $\phi \in \operatorname{Im}\sigma^{\mathsf{tr}}$. Indeed, using $\{0\} \times B_\epsilon(-\xi) \subseteq \bar{C}$, one gets the estimate $F(\phi) \geq -\xi^{\mathrm{tr}}\phi + \text{ess sup}_{\beta\in B_\epsilon(-\xi)}\,\beta^{\mathrm{tr}}(z-\phi) = -\xi^{\mathrm{tr}}z + \epsilon\,|z-\phi|$, which clearly implies coercivity of F. Hence by [ET99, Chapter II, Proposition 1.2], the function F admits a minimizer in $\operatorname{Im}\sigma^{\mathrm{tr}}$. By extending the arguments to random and time-dependent parameters, existence of $\bar{\phi} \in \Phi$ satisfying $f^{\bar{\phi}}(t, Z_t, U_t) = \text{ess inf}_{\phi\in\Phi} f^{\phi}(t, Z_t, U_t)$ follows by standard measurable selection arguments. Recall by Lemma 2.18 that it holds in particular that $f^{\bar{\phi}}(t, Z_t, U_t) = \text{ess inf}_{\phi\in\Phi} f^{\phi}(t, Z_t, U_t) = f(t, Z_t, U_t)$ for all $t \in [0, T]$. As a consequence of uniqueness of the solution of the JBSDE (2.18), we obtain that $Y^{\bar{\phi}} = Y$. Now by Lemma 2.17 and part a) of Theorem 2.16 follows $\pi^u_t(X) = Y_t = Y^{\bar{\phi}}_t = \rho_t\Big(X - \int_t^T \bar{\phi}^{\mathrm{tr}}_s d\widehat{W}_s\Big),\ t \in [0, T]$. To conclude that (2.23) holds, it remains to show that for any $\phi \in \Phi$ one has $Y \leq Y^{\phi}$. Let ϕ be in Φ. Then

$$\begin{aligned} f^{\phi}_t(Z^{\phi}_t, U_t) - f^{\phi}_t(Z^{\phi}_t, U^{\phi}_t) =& (Z^{\phi}_t - \phi_t)^{\mathrm{tr}}\big(\tilde{\beta}_t(Z^{\phi}-\phi, U) - \tilde{\beta}_t(Z^{\phi}-\phi, U^{\phi})\big) \\ &+ \int_E U_t(e)\tilde{\gamma}_t(Z^{\phi}-\phi, U)(e)\lambda_t(de) \\ &- \int_E U^{\phi}_t(e)\tilde{\gamma}_t(Z^{\phi}-\phi, U^{\phi})(e)\lambda_t(de). \end{aligned} \tag{2.69}$$

By part b) of Lemma 2.14, the couple $\big(\tilde{\gamma}_t(Z^{\phi}-\phi, U^{\phi}), \tilde{\beta}_t(Z^{\phi}-\phi, U^{\phi})\big)$ is the maximizer of the expression $(Z^{\phi}_t - \phi_t)^{\mathrm{tr}}\beta_t + \int_E U^{\phi}_t(e)\gamma_t(e)\lambda_t(de)$ over all $(\gamma_t, \beta_t) \in \bar{C}_t$. Now by the fact that $\big(\tilde{\gamma}_t(Z^{\phi}-\phi, U), \tilde{\beta}_t(Z^{\phi}-\phi, U)\big) \in \bar{C}_t$ holds, it follows

$$\begin{aligned} (Z^{\phi}_t - \phi_t)^{\mathrm{tr}}\tilde{\beta}_t(Z^{\phi}-\phi, U^{\phi}) &+ \int_E U^{\phi}_t(e)\tilde{\gamma}_t(Z^{\phi}-\phi, U^{\phi})(e)\lambda_t(de) \\ &\geq (Z^{\phi}_t - \phi_t)^{\mathrm{tr}}\tilde{\beta}_t(Z^{\phi}-\phi, U) + \int_E U^{\phi}_t(e)\tilde{\gamma}_t(Z^{\phi}-\phi, U)(e)\lambda_t(de). \end{aligned}$$

Using this inequality in (2.69) implies

$$f^{\phi}_t(Z^{\phi}_t, U_t) - f^{\phi}_t(Z^{\phi}_t, U^{\phi}_t) \leq \int_E \tilde{\gamma}_t(Z^{\phi}-\phi, U)(e)(U_t(e) - U^{\phi}_t(e))\lambda_t(de). \tag{2.70}$$

By Assumption 2.13, $(\tilde{\gamma}_t(Z^{\phi}-\phi, U), \tilde{\beta}_t(Z^{\phi}-\phi, U))$ is bounded in $\mathbb{R}^n \times L^2(\lambda_t(\omega))$ uniformly in (t, ω). With this at hand, one shows using Proposition 2.31 and following the arguments in the proof of Proposition 2.3 that the stochastic exponential $\mathcal{E}\big(\tilde{\beta}(Z^{\phi}-\phi, U)\cdot W + \tilde{\gamma}(Z^{\phi}-\phi, U) * \widetilde{\mu}\big)$ is a uniformly integrable martingale. Now applying Proposition 2.6 (plus Remark 2.7) to the JBSDEs with parameters $(f_1, X_1) := (f, X)$ and $(f_2, X_2) := (f^{\phi}, X)$ yields $Y_t \leq Y^{\phi}_t$, $t \in [0, T]$.

To show the second claim of the theorem, let $Q^{\gamma,\beta} \in \mathcal{P}^{\mathsf{ngd}}$. The tracking error is given by $R^{\bar{\phi}}(X) = \pi^u_t(X) - \pi^u_0(X) - \bar{\phi}\cdot\widehat{W}_t$. Using part a) of Theorem 2.16 and a change of measure to $Q^{\gamma,\beta}$, one obtains for all $t \in [0, T]$ that

$$\begin{aligned} -dR^{\bar{\phi}}_t(X) &= f(t, Z_t, U_t)dt - Z_t dW_t - \int_E U_t(e)\widetilde{\mu}(dt, de) + \bar{\phi}^{\mathrm{tr}}_t d\widehat{W}_t \\ &= \Big(f(t, Z_t, U_t) + \xi^{\mathrm{tr}}_t\bar{\phi}_t - (Z_t - \bar{\phi}_t)^{\mathrm{tr}}\beta_t - \int_E U_t(e)\gamma_t(e)\lambda_t(de)\Big)dt \\ &\qquad - (Z_t - \bar{\phi}_t)dW^{Q^{\gamma,\beta}}_t - \int_E U_t(e)\widetilde{\mu}^{Q^{\gamma,\beta}}(dt, de). \end{aligned}$$

Since by Lemma 2.18 we have $f^{\bar{\phi}}(t, Z_t, U_t) = \text{ess inf}_{\phi\in\Phi} f^{\phi}(t, Z_t, U_t) = f(t, Z_t, U_t)$, then the finite variation part under $Q^{\gamma,\beta}$ of $R^{\bar{\phi}}(X)$ is non-decreasing and vanishes for $(\gamma, \beta) = (\gamma^*, \beta^*)$. By Assumption 2.13 $\|(\gamma_t(\omega), \beta_t(\omega))\|_{L^2(\lambda_t(\omega))\times\mathbb{R}^n}$ is bounded uniformly in (t, ω), hence $dQ^{\gamma,\beta}/dP = \mathcal{E}(\beta \cdot W + \gamma * \widetilde{\mu}) \in \mathcal{S}^2$. Since $R^{\bar{\phi}}(X) \in \mathcal{S}^2$, then Hölder's inequality implies $R^{\bar{\phi}}(X) \in \mathcal{S}^1(Q^{\gamma,\beta})$, and therefore $R^{\bar{\phi}}(X)$ is a $Q^{\gamma,\beta}$-supermartingale and a martingale under $Q^* = Q^{\gamma^*,\beta^*}$. □

Proof of Lemma 2.22. One rewrites $\widetilde{C}_t(\omega) = \{(\gamma, \beta) \in H_t(\omega) : \ G(\gamma, \beta) \in I_t(\omega)\}$, where the map $G : L^2(\lambda) \times \mathbb{R}^n \to \mathbb{R}$ is defined by $G(\gamma, \beta) := \|\gamma\|^2_{L^2(\lambda)} + |\beta|^2$, and I and H are closed-convex-valued correspondences with values

$$I_t(\omega) = [0, K_t^2(\omega)] \subseteq \mathbb{R} \quad \text{and} \quad H_t(\omega) = \left\{\gamma \in L^2(\lambda) : \ \gamma \geq -\sqrt{\zeta_t(\omega)}\right\} \times \mathbb{R}^n \subseteq L^2(\lambda) \times \mathbb{R}^n.$$

Since K is a predictable process, then I is predictable (and in particular $\overline{\mathcal{P}}$-measurable). Since $G(\cdot)$ is continuous, then applying two times [AF90, Theorem 8.2.9] (noting that $L^2(\lambda)$ is a separable Hilbert space, hence a Polish space), one obtains first that H is $\overline{\mathcal{P}}$-measurable and then that $\widetilde{C}$ is $\overline{\mathcal{P}}$-measurable (since I is). □

Proof of Theorem 2.26. Without loss of generality, we argue for $X \geq 0$, since otherwise one can use a translation argument with $X + \|X\|_\infty \geq 0$.

Part 1: For $t \in [0, T]$, since $C_t^k(\omega) \subseteq C_t^{k+1}(\omega) \subseteq C_t(\omega)$ for all $k \in \mathbb{N}$, then $\pi_t^{u,k}(X) \leq \pi_t^{u,k+1}(X) \leq \pi_t^u(X)$, for any $k \in \mathbb{N}$. Since X is bounded, then the monotone a.s. limit $J_t := \lim_{k\nearrow\infty} \pi_t^{u,k}(X)$ is finite and $J_t \leq \pi_t^u(X)$. It remains to show that $\pi_t^u(X) \leq J_t$ holds. To this end we show that J is a càdlàg Q-supermartingale for all $Q \in \mathcal{Q}^{\text{ngd}}$ and use Lemma 2.25. First J is a càdlàg $Q^{\gamma,\beta}$-supermartingale for any $Q^{\gamma,\beta} \in \mathcal{Q}^{\text{ngd}}$ with Girsanov kernel (γ, β) satisfying $\|\gamma\|^2_{L^2(\lambda_t)} + |\beta|^2 \leq c, \ t \in [0, T]$, for some constant $c > 0$. Indeed for such a measure $Q^{\gamma,\beta}$, there exists $k_0 \in \mathbb{N}$ such that $(\gamma, \beta) \in C^k$ for all $k \geq k_0$. Since $J_t = \lim_{k\nearrow\infty, k\geq k_0} \pi_t^{u,k}(X)$ and $\pi^{u,k}(X)$ is a bounded càdlàg $Q^{\gamma,\beta}$-supermartingale for every $k \geq k_0$, then J is a càdlàg $Q^{\gamma,\beta}$-supermartingale as the increasing limit of càdlàg $Q^{\gamma,\beta}$-supermartingales of class D (see e.g. [Doo01, Section 2.IV.4]). This is in particular valid when $Q^{\gamma,\beta} \in \mathcal{Q}_k^{\text{ngd}}$ for some $k \in \mathbb{N}$. Now for an arbitrary $Q^{\gamma,\beta} \in \mathcal{Q}^{\text{ngd}}$, i.e. generally satisfying $(\gamma, \beta = -\xi + \eta) \in C$, define the sequence $(\gamma^k, \eta^k)_{k\in\mathbb{N}}$ with $(\gamma_t^k, \eta_t^k) := (\gamma_t, \eta_t)\mathbb{1}_{\left\{\|(\gamma_t,\eta_t)\|_{L^2(\lambda_t)\times\mathbb{R}^n} \leq k\right\}}, \ t \in [0, T]$. Then $(\gamma^k, \beta^k = -\xi + \eta^k) \in C^k$ and hence $Q^k := Q^{\gamma^k,\beta^k} \in \mathcal{Q}_k^{\text{ngd}}$. Moreover $\lim_k \gamma^k = \gamma, \ P \otimes \lambda \otimes dt$-a.e. and $\lim_k \eta^k = \eta, \ P \otimes dt$-a.e. By the above argument, since ξ and X are bounded, then J is a bounded càdlàg $\widehat{Q}$-supermartingale and hence admits a Doob-Meyer decomposition which, using weak predictable representation property of $(\widehat{W}, \widetilde{\mu})$ under $(\widehat{Q}, \mathbb{F})$ (see e.g. [HWY92, Theorem 3.22] or Part 2. in Example 1.1 of Chapter 1) and the fact that $\nu^{\widehat{Q}} = \nu$, reads $J = J_0 + Z \cdot \widehat{W} + U * \widetilde{\mu} - A$, for $(Z, U) \in \mathcal{H}^2(\widehat{Q}) \times \mathcal{H}^2_\nu$ and A a non-decreasing predictable processes with $A_0 = 0$ and A_T being $\widehat{Q}$-integrable since J is bounded (cf. [DM82, Inequality (15.1), Section VII.15]). By a change of measure from $\widehat{Q}$ to Q^k on one hand and to $Q^{\gamma,\beta}$ on the other hand, one rewrites

$$J = J_0 + Z \cdot W^{Q^k} + U * \widetilde{\mu}^{Q^k} + \int_0^{\cdot} \Big(Z_t^{\mathrm{tr}}\eta_t^k + \int_E U_t(e)\gamma_t^k(e)\lambda_t(de)\Big)dt - A, \tag{2.71}$$

and

$$J = J_0 + Z \cdot W^{Q^{\gamma,\beta}} + U * \widetilde{\mu}^{Q^{\gamma,\beta}} + \int_0^{\cdot} \Big(Z_t^{\mathrm{tr}}\eta_t + \int_E U_t(e)\gamma_t(e)\lambda_t(de)\Big)dt - A. \tag{2.72}$$

Now since $(\gamma^k, \eta^k) \in C^k$ then J is a bounded càdlàg Q^k-supermartingale for any $k \in \mathbb{N}$. Hence from (2.71), it follows that $dA_t \geq \Big(Z_t^{\mathrm{tr}}\eta_t^k + \int_E U_t(e)\gamma_t^k(e)\lambda_t(de)\Big)\,dt$. Taking the limit as k goes to ∞ and using the dominated convergence theorem (since $\left|\eta^k\right| \leq |\eta|$ and $\left|\gamma^k\right| \leq |\gamma|$ a.s.) one obtains $dA_t \geq \left(Z_t^{\mathrm{tr}}\eta_t + \int_E U_t(e)\gamma_t(e)\lambda_t(de)\right)dt$, $t \in [0,T]$ and hence the process $A - \int_0^{\cdot}\left(Z_t^{\mathrm{tr}}\eta_t + \int_E U_t(e)\gamma_t(e)\lambda_t(de)\right)dt$ is non-decreasing and in particular non-negative. Now since X is non-negative, so is J and this implies from (2.72) that $J_0 + Z \cdot W^{Q^{\gamma,\beta}} + U * \widetilde{\mu}^{Q^{\gamma,\beta}}$ is a non-negative local $Q^{\gamma,\beta}$-martingale and is therefore a $Q^{\gamma,\beta}$-supermartingale. Finally, $Q^{\gamma,\beta}$-integrability of $\int_0^T \left(Z_t^{\mathrm{tr}}\eta_t + \int_E U_t(e)\gamma_t(e)\lambda_t(de)\right)dt - A_T$ follows, by boundedness of J, and thus J is a $Q^{\gamma,\beta}$-supermartingale.

Part 2: For $k \in \mathbb{N}$, the process $\pi^{u,k}$ can be seen as the good-deal bound associated to the correspondence C^k satisfying the hypotheses of Theorem 2.16, which implies the required result, after using the a-priori estimates of [Bec06, Proposition 3.3] to obtain $Y \in \mathcal{S}^\infty$.

Part 3: For any $k \geq \|\xi\|_\infty$ holds $\widehat{Q} \in \mathcal{Q}_k^{\mathrm{ngd}} \subset \mathcal{Q}^{\mathrm{ngd}}$. Hence by Lemma 2.25, $\pi^u(X)$ and $\pi^{u,k}(X)$ (for $k \geq \|\xi\|_\infty$) are bounded càdlàg $\widehat{Q}$-supermartingales since X is bounded. Thus they admit Doob-Meyer decompositions (2.48) and (2.49) respectively. Now since the triple $(\pi^{u,k}(X), Z^k, U^k)$ solves the JBSDE (2.47), one obtains that A^k satisfies (2.50). Moreover $(Z,U) \in \mathcal{H}^2(\widehat{Q}) \times \mathcal{H}^2_\nu(\widehat{Q})$ and $A_T, A_T^k \in L^2(\widehat{Q})$ follows from arguments in the proof of Part 1.

Part 4: From Part 3., that A_u^k converges to A_u weakly in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$ as $k \to \infty$ for all $u \in [0,T]$ follows from [DM82, Theorem VII.18 and subsequent remark]. This applies since the sequence $\left(\pi^{u,k}(X)\right)_{k \geq \|\xi\|_\infty}$ is uniformly bounded by $\|X\|_\infty$ (and hence uniformly of class D), and therefore Part 1. and dominated convergence imply that $\pi_u^{u,k}(X)$ converges to $\pi_u^u(X)$ in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$ as $k \to \infty$ for all $u \in [0,T]$. Furthermore the convergences of the sequences $(\pi_u^{u,k}(X))_{k\in\mathbb{N}}$ and $(A_u^k)_{k\in\mathbb{N}}$ imply that $Z^k \cdot \widehat{W}_u + \int_0^u \int_E U_s^k(e)\widetilde{\mu}(ds,de)$ converges to $Z \cdot \widehat{W}_u + \int_0^u \int_E U_s(e)\widetilde{\mu}(ds,de)$ weakly in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$ for all $u \in [0,T]$. By the weak predictable representation property of $(\widehat{W}, \widetilde{\mu})$ under $\widehat{Q}$, strong orthogonality and isometry, the required result follows.

Part 5: By Lemma 2.10, $\pi^u(X)$ is a Q-supermartingale with terminal value X, for all $Q \in \mathcal{Q}^{\mathrm{ngd}}$. Since by assumption $\bar{Q}$ is in the L^1-closure of $\mathcal{Q}^{\mathrm{ngd}}$, then $\pi^u(X)$ is also a $\bar{Q}$-supermartingale with terminal value X. This together with $\pi_0^u(X) = E^{\bar{Q}}[X]$ implies that $\pi_\cdot^u(X)$ has constant $\bar{Q}$-expectation and is therefore a $\bar{Q}$-martingale. Further, quasi-left-continuity of $\pi^u(X)$ is clear since $\nu \ll \lambda \otimes dt$ implies $\nu^{\bar{Q}} \ll \lambda \otimes dt$ by part a) of Lemma 2.1. Since $X \in L^\infty$, then to show that $\pi^{u,k}(X)$ converges to $\pi^u(X)$ in $\mathcal{S}^p(\widehat{Q})$ for any $p \in [1,\infty)$, it suffices by dominated convergence to show that $\sup_{t\in[0,T]}|\pi_t^{u,k}(X) - \pi_t^u(X)|$ converges to 0 in probability. By part 1. we know that $\pi_t^{u,k}(X) \nearrow \pi_t^u(X)$ a.s. for all $t \in [0,T]$. Moreover, denoting by ${}^pY$ the predictable projection of an integrable process Y relative to the filtration $\mathbb{F}$, it holds that ${}^p\pi^{u,k}(X)_t \nearrow {}^p\pi^u(X)_t$ for any $t \in [0,T]$. Recall that for every uniformly integrable martingale M holds ${}^pM = M_-$, and that for a predictable process K one has ${}^pK = K$. Now because $\pi^{u,k}(X)$ is a bounded $\widehat{Q}$-supermartingale and A^k is continuous, then holds ${}^p\pi^{u,k}(X)_t = \pi_{t-}^{u,k}(X)$. Moreover by quasi-left-continuity of $\pi^u(X)$, one has that A is a continuous process (by [HWY92, Theorem 5.50]) and hence ${}^p\pi^u(X)_t = \pi_{t-}^u(X)$. As a consequence one has $\pi_{t-}^{u,k}(X) \nearrow \pi_{t-}^u(X)$ a.s. for all $t \in [0,T]$. Now by the extended Dini's Lemma in [DM82, Page 185] uniform convergence in time follows, i.e. $\sup_{t\in[0,T]}|\pi_t^{u,k}(X) - \pi_t^u(X)| \searrow 0$. To prove the remaining claims, note that for all $k \in \mathbb{N}$ holds $E^{\widehat{Q}}[A_T^k] \le E^{\widehat{Q}}[|\pi_T^{u,k}(X) - \pi_0^{u,k}(X)|] \le 2\|X\|_\infty$, which implies that $\sup_{k\in\mathbb{N}} E^{\widehat{Q}}[\int_0^T |dA_t^k|] < \infty$. Finally by [BP90, Corollary 2], the required claims follow. $\square$

Proof of Proposition 2.29.

Part 1: If $Q^{\gamma,\beta} \in \mathcal{M}^e$, then by Lemma 2.8 it follows that $\beta = -\xi + \eta$ with the required properties, and moreover $-\log \Gamma_t^{\gamma,\beta} = -M_t + \frac{1}{2}\langle M^c\rangle_t - \sum_{s\le t}(\log(1+\Delta M_s) - \Delta M_s)$, holds for $t \in [0,T]$. Define the process V by $V_t := \sum_{s\le t}(\log(1+\Delta M_s) - \Delta M_s)$. Since $E[-\log \Gamma_T^{\gamma,\beta}] < \infty$ then by Proposition 2.27 the process $-\log \Gamma^{\gamma,\beta}$ is a P-submartingale of class (D) with a Doob-Meyer decomposition $-\log \Gamma^{\gamma,\beta} = N^{\gamma,\beta} + A^{\gamma,\beta}$. Hence M, $\langle M^c\rangle = \int_0^\cdot |\beta_s|^2\, ds$ and V are locally P-integrable with

$$-V_t = (-\log(1+\gamma) + \gamma) * \mu_t = g(1+\gamma) * \mu_t = g(1+\gamma) * \widetilde{\mu}_t + g(1+\gamma) * \nu_t,$$

where the third equality is obtained from [JS03, Proposition II.1.28] and $g \ge 0$. Hence one has $-\log \Gamma_t^{\gamma,\beta} = -M_t + g(1+\gamma) * \widetilde{\mu}_t + \frac{1}{2}\langle M^c\rangle_t + g(1+\gamma) * \nu_t$. Now because $M = M^c + M^d$ with $M^c = \beta \cdot W$ and $M^d = \gamma * \widetilde{\mu}$, it follows that

$$-\log \Gamma_t = \underbrace{-\beta \cdot W_t - \log(1+\gamma) * \widetilde{\mu}_t}_{:=N_t^{\gamma,\beta}} + \underbrace{\int_0^t \Big(\frac{1}{2}|\beta_s|^2 + \int_E g(1+\gamma_s(e))\lambda_s(de)\Big) ds}_{:=A_t^{\gamma,\beta}},$$

where the process $\log(1+\gamma) * \widetilde{\mu}$ is locally P-integrable thanks to the local P-integrability of V and $\gamma * \widetilde{\mu}$, and an application of [JS03, Proposition II.1.28].

Part 2: For $Q \in \mathcal{Q}^{\mathsf{ngd}} \subset \mathcal{M}^e$, Proposition 2.28 implies that $(\kappa^Q - \upsilon)(B) \le 0$, for any $B \in \mathcal{P}$. Choosing $B = \left\{\frac{1}{2}|\beta|^2 + \int_E g(1+\gamma_\cdot(e))\zeta_\cdot(e)\lambda(de) > \frac{1}{2}K^2\right\}$, the Fubini's theorem (see e.g. [Coh13, Proposition 5.2.1]) gives $B \in \mathcal{P}$ and hence $(\kappa^Q - \upsilon)(B) \le 0$ holds, which implies by definition of κ^Q and υ that B is a $P\otimes dt$-nullset. Thus $\frac{1}{2}|\beta|^2+\int_E g(1+\gamma.(e))\zeta.(e)\lambda(de) \le \frac{1}{2}K^2$ holds, which is equivalent to (2.56), since by Part 1. one has $\Pi_t(\beta_t)=\xi_t$ and $\Pi_t^\perp(\beta_t)=\eta_t$.

Part 3: Let $\gamma>-1$ be a $\widetilde{\mathcal{P}}$-measurable and $\widetilde{\mu}$-integrable function, and let β be a predictable process with $\Pi_t(\beta_t)=\xi_t$ and $\Pi_t^\perp(\beta_t)=\eta_t,\ t\in[0,T]$, such that (γ,β) satisfies the inequality $\frac{1}{2}|\beta|^2+\int_E g(1+\gamma.(e))\zeta.(e)\lambda(de) \le \frac{1}{2}K^2$. Then $\int_E g(1+\gamma.(e))\zeta.(e)\lambda(de) \le \frac{1}{2}K^2$ and $|\beta|^2\le K^2$. By boundedness of K, the couple (γ,β) defines from Proposition 2.3 the Girsanov kernels of a measure $Q\sim P$. That $Q\in\mathcal{M}^e$ follows from Lemma 2.8. For such a measure Q the integrability condition on (γ,β) directly implies from the last claim of Proposition 2.27 that (2.54) is satisfied; hence $Q\in\mathcal{Q}^{\mathsf{ngd}}$. □

Proof of Lemma 2.30. From the discussion preceding the statement of the lemma, one already has $\widetilde{C}_t\subseteq\bigcap_{l\in\mathbb{N}}\widetilde{C}_t^l,\ t\in[0,T]$. Now let $t\le T$ and $(\gamma,\beta)\in\bigcap_{l\in\mathbb{N}}\widetilde{C}_t^l$, i.e. $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma,\beta)\in\bar{C}_t^l$, for all $l\in\mathbb{N}$. It is to be shown that $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma,\beta)\in\bar{C}_t$. Define the sequence $(\gamma^k,\beta^k),\ k\in\mathbb{N}$ such that $\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k := (\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma)\vee(-1+1/k)$ and $\beta^k:=\beta,\ k\in\mathbb{N}$. One has for all $k\in\mathbb{N}$ that $|\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k|\le|\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma|$ (since $\gamma\ge-\sqrt{\zeta_t}$) and so dominated convergence implies that $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k,\beta^k)$ converges to $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma,\beta)$ in $L^2(\lambda_t)\times\mathbb{R}^n$ as $k\to\infty$. To conclude that $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma,\beta)\in\bar{C}_t$ it remains to show that $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k,\beta)\in C_t$ holds for any $k\in\mathbb{N}$. For this purpose, one has to show that $\|2g(1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k)\|_{L^1(\lambda_t)}+|\beta|^2\le K_t^2$ holds. Note that $g(1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k)=g^k(1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma)$ λ-a.e. since $1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k$ either takes the value $1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma$ when $1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma\ge1/k$ or $1/k$ when $1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma\le1/k$. This implies that for all $k\in\mathbb{N},\ \|2g(1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma^k)\|_{L^1(\lambda_t)}=\|2g^k(1+\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma)\|_{L^1(\lambda_t)}$. Since $(\zeta_t^{-1/2}\mathbb{1}_{\{\zeta_t>0\}}\gamma,\beta)\in\bar{C}_t^k$, for all $k\in\mathbb{N}$, this concludes the first claim of the lemma.

To prove the second claim, recall that for any $l\in\mathbb{N}$ the function g^l satisfies $g^l(1+y)\le$ Const $|y|^2$ for all $y\ge-1$ for some Const >0. We first show that the map $G^l:[0,T]\times\Omega\times L^2(\lambda)\times\mathbb{R}^n\to\mathbb{R}$ defined by $G^l(t,\omega,\gamma,\beta):=|\beta|^2+\|2g^l(1+\zeta_t^{-1/2}(\omega)\gamma)\|_{L^1(\lambda_t(\omega))}$ is continuous in $(\gamma,\beta)\in L^2(\lambda)\times\mathbb{R}^n$. For this purpose, it suffices to show that for a sequence $(\gamma^n)_n\subset L^2(\lambda)$ which converges in $L^2(\lambda)$ to γ, the sequence of integrals $\int_E g^l(1+\zeta_t^{-1/2}\gamma^n(e))\lambda_t(de)$ converges to $\int_E g^l(1+\zeta_t^{-1/2}\gamma(e))\lambda_t(de)$. If the measure λ is finite, this follows straightforwardly from the Lipschitz property of the function g^l. For infinite λ, note that there exist by [AB06, Theorem 13.6] a subsequence $(\gamma^{n_k})_k$ of $(\gamma^n)_n$ and a function $\psi\in L^2(\lambda)$ such that $|\gamma^{n_k}|\le\psi$ λ-a.e. and γ^{n_k} converges λ-a.e. to γ. Renaming the subsequence if necessary we can assume that $\int_E g^l(1+\zeta_t^{-1/2}\gamma^{n_k}(e))\lambda_t(de)$ converges to $\limsup_n\int_E g^l(1+\zeta_t^{-1/2}\gamma^n(e))\lambda_t(de)$ as $k\to\infty$. Dominated convergence implies that $\int_E g^l(1+\zeta_t^{-1/2}\gamma^{n_k}(e))\lambda_t(de)$ converges to $\int_E g^l(1+\zeta_t^{-1/2}\gamma(e))\lambda_t(de)$, and therefore $\limsup_n\int_E g^l(1+\zeta_t^{-1/2}\gamma^n(e))\lambda_t(de)=\int_E g^l(1+\zeta_t^{-1/2}\gamma(e))\lambda_t(de)$. By similar arguments one can also show that $\liminf_n \int_E g^l(1+\zeta_t^{-1/2}\gamma^n(e))\lambda_t(de) = \int_E g^l(1+\zeta_t^{-1/2}\gamma(e))\lambda_t(de)$, and this concludes the continuity of G^l. Now by Fubini's theorem ([Coh13, Proposition 5.2.1]), the map G^l is in addition $\overline{\mathcal{P}}$-measurable in $(t,\omega) \in [0,T]\times\Omega$ since ζ is $\widetilde{\mathcal{P}}$-measurable and g^l is continuous. Hence G^l is a Carathéodory map and following the same arguments as those of the proof of Lemma 2.22 one shows that $\widetilde{C}^l$ is $\overline{\mathcal{P}}$-measurable for any $l \in \mathbb{N}$. Now $\overline{\mathcal{P}}$-measurability of $\widetilde{C}$ follows by [AF90, Theorem 8.2.4]. $\square$

3. Hedging under generalized good-deal bounds and drift uncertainty

In this chapter, we study good-deal valuation and hedging as in Chapter 2 but in a Brownian setting (i.e. continuous filtration), focusing on constructive examples with closed-form solutions and on robustness with respect to uncertainty (ambiguity) about the market price of risk (excess returns) of hedging assets. We describe robust good-deal bounds and hedging strategies by solutions to standard BSDEs, and show that robust good-deal hedging is equivalent to risk-minimization (with respect to a specific no-good-deal pricing measure depending on the claim to be hedged) if uncertainty is very large. Section 3.1 formulates a framework for good-deal constraints which are described by predictable correspondences sufficiently general for all later sections to incorporate the natural radial (Sharpe ratio) constraints that are predominant in the good deal literature, but also extensions to ellipsoidal constraints. Section 3.2 studies hedging strategies and provides new examples with closed-form formulas for good-deal bounds and hedging strategies. In the presence of model uncertainty, good deal bounds and hedging strategies that are robust with respect to uncertainty are derived the Section 3.3, with the link to risk-minimization made more precise. In Appendix 3.4 we provide statements of intermediary results and proofs that are omitted in the main body of the chapter.

3.1 Mathematical framework and preliminaries

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with time horizon $T < \infty$; the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by an n-dimensional Brownian motion W, augmented with P-null-sets, satisfying the usual conditions. Let $\mathcal{F} = \mathcal{F}_T$. Inequalities between random variables (processes) are meant to hold almost everywhere with respect to P (resp. $P \otimes dt$). For stopping times $\tau \leq T$, the conditional expectation given $\mathcal{F}_\tau$ under a probability measure Q is denoted by $E^Q_\tau[\,\cdot\,]$. We write $E_\tau = E^P_\tau$ if there is no ambiguity about P. $L^p(\mathbb{R}^m, Q)$, $p \in [1, \infty)$, (or $L^\infty(\mathbb{R}^m, Q)$) denotes the space of $\mathcal{F}_T$-measurable $\mathbb{R}^m$-valued random variables X with $\|X\|^p_{L^p(Q)} = E^Q[|X|^p] < \infty$ (resp. X Q-essentially bounded). $\mathcal{P}$ denotes the predictable σ-field on $[0, T] \times \Omega$. Stochastic integrals of predictable integrands H with respect to semimartingales S are denoted $H \cdot S = \int_0^\cdot H^{\mathrm{tr}}_t dS_t$. Let $\mathcal{H}^p(\mathbb{R}^m, Q)$ denote the space of predictable $\mathbb{R}^m$-valued processes Z with $\|Z\|^p_{\mathcal{H}^p(Q)} = E^Q\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right] < \infty$, and $\mathcal{S}^p(Q)$ that of càdlàg semimartingales Y with $\|Y\|_{\mathcal{S}^p(Q)} = \left\|\sup_{t \leq T} |Y_t|\right\|_{L^p(Q)} < \infty$ If the dimension is clear, we just write $L^p(Q)$ and $\mathcal{H}^p(Q)$, and if $Q = P$ just L^p, $\mathcal{H}^p$ and $\mathcal{S}^p$, for $p \in [1, \infty]$. The Euclidean norm of a matrix $M \in \mathbb{R}^{n \times d}$ is $|M| := (\mathrm{Tr}\, MM^{\mathrm{tr}})^{1/2}$ and its usual operator norm is denoted by $\|M\|$.

We will make use of classical theory of BSDEs [PP90, EPQ97]. BSDEs are stochastic differential equation of the type

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t^{\mathrm{tr}} dW_t, \text{ for } t \leq T, \text{ and } Y_T = X, \tag{3.1}$$

where the terminal condition X is an $\mathcal{F}_T$-measurable random variable and the generator function $f : \Omega \times [0,T] \times \mathbb{R}^{1+n} \to \mathbb{R}$ a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+n})$-$\mathcal{B}(\mathbb{R})$-measurable function. They are well established in mathematical economics. A pair (f, X) constitutes standard parameters (also called data) for a BSDE (3.1) if $X \in L^2$, $f(\cdot, 0, 0)$ is in $\mathcal{H}^2$ and f is uniformly Lipschitz in y and z, i.e. there exists $L < \infty$ such that $|f(\omega, t, y, z) - f(\omega, t, y', z')| \leq L(|y - y'| + |z - z'|)$ holds for all t, y, z, y', z'. A solution of the BSDE (3.1) is a couple (Y, Z) of processes such that Y is real-valued continuous, adapted, and Z is $\mathbb{R}^n$-valued predictable and satisfies $\int_0^T |Z_t|^2 dt < \infty$. For standard parameters (f, X) there exists a unique solution $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$ to the BSDE (3.1), [EPQ97, Theorem 2.1]. Let us refer to BSDEs with standard parameters as *classical* and to the solution to such BSDEs as *standard*. A comparison theorem [EPQ97, Proposition 3.1] is very useful for optimal control problems stated in terms of classical BSDEs: Given standard BSDE solutions $(Y, Z), (Y^a, Z^a)_{a \in A}$ for a family of standard parameters $(f, X), (f^a, X^a)_{a \in A}$, if there exists $\bar{a} \in A$ such that $f(t, Y_t, Z_t) = \operatorname*{ess\,inf}_{a \in A} f^a(t, Y_t, Z_t) = f^{\bar{a}}(t, Y_t, Z_t)$, $t \leq T$, and $X = \operatorname*{ess\,inf}_{a \in A} X^a = X^{\bar{a}}$, then $Y_t = \operatorname*{ess\,inf}_{a \in A} Y_t^a = Y_t^{\bar{a}}$ holds for all $t \leq T$.

Section 3.1.1 will specify a financial market with d risky assets whose discounted price processes S^i $(i \leq d)$ with respect to a fixed numéraire asset (with unit price $S^0 = 1$) are non-negative locally bounded semimartingales. The set of equivalent local martingale measures (risk neutral pricing measures) is denoted by $\mathcal{M}^e := \mathcal{M}^e(S)$ and we assume $\mathcal{M}^e \neq \emptyset$, i.e. there is no free lunch with vanishing risk in the sense of [DS94]. The market is incomplete with $\mathcal{M}^e$ being of infinite cardinality if $d < n$. We will define generalized good-deal bounds by using abstract predictable correspondences C defined on $[0,T] \times \Omega$ with non-empty compact and convex values $C_t(\omega) \subset \mathbb{R}^n$, with predictability in the sense of [Roc76], i.e. for each closed set $F \subset \mathbb{R}^n$, the set $C^{-1}(F) := \{(t, \omega) \in [0,T] \times \Omega : \ C_t(\omega) \cap F \neq \emptyset\}$ is predictable. More specific examples, e.g. for ellipsoidal constraints, will exhibit (semi)explicit solutions for optimizers. We write $C : [0,T] \times \Omega \rightsquigarrow \mathbb{R}^n$ with "$\rightsquigarrow$" to emphasize that C is a set-valued mapping, and $\lambda \in C$ to mean that the predictable function λ is a selection of C, i.e. $\lambda_t(\omega) \in C_t(\omega)$ holds on $[0,T] \times \Omega$. In the sequel a *standard* correspondence will refer to a predictable one, whose values are non-empty, compact and convex. Let $C : [0,T] \times \Omega \rightsquigarrow \mathbb{R}^n$ be a fixed standard correspondence with $0 \in C$. The set $\mathcal{Q}^{\mathrm{ngd}} := \mathcal{Q}^{\mathrm{ngd}}(S)$ of (equivalent) no-good-deal measures is given by

$$\mathcal{Q}^{\mathrm{ngd}}(S) := \left\{ Q \in \mathcal{M}^e \,\middle|\, dQ/dP = \mathcal{E}(\lambda \cdot W), \lambda \text{ predictable, bounded, } \lambda \in C \right\}. \tag{3.2}$$

In the definition (3.2) and in subsequent definitions of sets of equivalent measures, we tacitly assume that Girsanov kernels λ are such that the stochastic exponentials $\mathcal{E}(\lambda \cdot W)$ are uniformly integrable martingales. For good-deal valuation and hedging results later, concrete assumptions (e.g. Assumption 3.3) ensure that such holds for all selections λ of C. We remark that for good time-consistency properties, good-deal constraints should be specified locally in time ([KS07b]). For contingent claims X in L^2, upper and lower good-deal valuation bounds

$$\pi_t^l(X) := \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X] \quad \text{and} \quad \pi_t^u(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X], \quad t\in[0,T]. \tag{3.3}$$

are defined over a suitable (yet abstract) set of no good deal pricing measures $\mathcal{Q}^{\mathrm{ngd}}$. Hence $\pi_t^u(X)$ (respectively $\pi_t^l(X)$) can be seen as the highest (lowest) valuation that does not permit too good deals to the seller (buyer). Since $\pi_\cdot^l(X) = -\pi_\cdot^u(-X)$, further analysis can be restricted to $\pi_\cdot^u(X)$. As mentioned already in the introduction, the definition (3.3) in itself could already be viewed as a robust representation in a sense (over Q's). For our purpose here however, the correspondence C and the respective set $\mathcal{Q}^{\mathrm{ngd}}$ of no good deal measures are (at first) given with respect to one objective real world measure P (cf. remarks after (3.4)). To be clear in our use of terminology, we will in the sequel restrict our use of terms *model uncertainty, ambiguity* or *robust hedging/valuation* to situations with Knightian model uncertainty about P. Note that the use of terminology in some literature (e.g. [Del12]) is different, where the terms may instead refer to representations like (3.3). Definition (3.2) implies that density processes of measures $Q\in\mathcal{Q}^{\mathrm{ngd}}$ are in $\mathcal{S}^p$, $p\in[1,\infty)$. Hence $X\in L^2 = L^2(P)\subset L^1(Q)$. In particular for $X\in L^\infty\subset L^2$, we will show (cf. Theorem 3.7 and Proposition 3.5) that $\pi_t^u(X) = \operatorname{ess\,sup}_{Q\in\overline{\mathcal{Q}^{\mathrm{ngd}}}} E_t^Q[X]$, where

$$\overline{\mathcal{Q}^{\mathrm{ngd}}} := \Big\{ Q\in\mathcal{M}^e \Big|\, dQ/dP = \mathcal{E}\left(\lambda\cdot W\right),\ \lambda \text{ predictable and } \lambda\in C \Big\}. \tag{3.4}$$

is a larger set than $\mathcal{Q}^{\mathrm{ngd}}$, containing measures with Girsanov kernels that are not necessarily bounded. We recall that for radial constraints C (like in (3.21) with $A\equiv \mathrm{Id}_{\mathbb{R}^n}$ and constant $h\in(0,\infty)$), common in the good-deal literature, one has a known financial justification. By a direct duality argument, one can see (e.g. [Bec09, Section 3] in a semimartingale framework) that any (arbitrage-free) extension $\bar{S} = (S, S')$ of the market S by derivative price processes $S' := E_t^Q[X]$ for contingent claims X (with $Q\in\mathcal{M}^e$, $X^-\in L^\infty$, $X^+\in L^1(Q)$) does permit only wealth processes $V>0$ from self-financing trading strategy (in $\bar{S}$) whose expected growth rates (log utilities) over any time period $0\le t<\tau\le T$ satisfy the (sharp) estimate $E_t^P\left[\log\frac{V_\tau}{V_t}\right] \le E_t^P\left[-\log\frac{Z_\tau}{Z_t}\right]$, where Z is the density process of Q. For $Q\in\mathcal{Q}^{\mathrm{ngd}}$ with radial constraint, this estimate is bounded by $h^2(\tau-t)/2$, ensuring a bound $h^2/2$ to expected growth rates (good deals) for *any market extension* (ideas going back at least to [CR00, CH02]).

For the good-deal bounds to have nice dynamic properties, multiplicative stability (m-stability) of the set of no-good-deal measures is important. M-stability of dominated families of probability measures in dual representations (like e.g. (3.3)) for dynamic coherent risk measures (see e.g. [ADE$^+$07]) ensures in particular time consistency (recursiveness) and has been studied in a general context by [Del06]. In economics, it is known as rectangularity [CE02]. A set $\mathcal{Q}$ of measures $Q \sim P$ is called m-stable if for all $Q^1, Q^2 \in \mathcal{Q}$ with density processes Z^1, Z^2 and for all stopping times $\tau \leq T$, the process $Z := I_{[0,\tau]} Z^1_\cdot + I_{]\tau,T]} Z^1_\tau Z^2_\cdot / Z^2_\tau$ is the density process of a measure in $\mathcal{Q}$, where $[0,\tau] := \{(t,\omega) \in [0,T] \times \Omega \mid t \leq \tau(\omega)\}$ denotes the stochastic interval and I_A is the indicator function on a set A. As noted in [Del06, Rem. 6], by closure this definition extends to sets of measures that are absolutely continuous but not necessarily equivalent; such is formally achieved by setting $Z^2_T / Z^2_\tau = 1$ on $\{Z^2_\tau = 0\}$. The role of m-stability shows in results due to [Del06], stated in Lemma 3.1, Part a); for details cf. [KS07b, Theorem 2.7] or [Bec09, Proposition 2.6]. Proof for part b) is provided in the appendix.

Lemma 3.1. *Let $\mathcal{Q}$ be a convex and m-stable set of probability measures $Q \sim P$ and $\pi^{u,\mathcal{Q}}_t(X) :=$ ess sup$_{Q \in \mathcal{Q}}$ $E^Q_t[X]$, for $X \in L^\infty$.*
a) There exists a càdlàg version Y of $\pi^{u,\mathcal{Q}}_\cdot(X)$ such that for all stopping times $\tau \leq T$, $Y_\tau =$ ess sup$_{Q \in \mathcal{Q}} E^Q_\tau[X] =: \pi^{u,\mathcal{Q}}_\tau(X)$. Moreover $\pi^{u,\mathcal{Q}}_\cdot(\cdot)$ has the properties of a dynamic coherent risk measure. It is recursive and stopping time consistent: For stopping times $\sigma \leq \tau \leq T$ holds $\pi^{u,\mathcal{Q}}_\sigma(X^1) = \pi^{u,\mathcal{Q}}_\sigma\left(\pi^{u,\mathcal{Q}}_\tau(X^1)\right)$, and $\pi^{u,\mathcal{Q}}_\tau(X^1) \geq \pi^{u,\mathcal{Q}}_\tau(X^2)$ for $X^1, X^2 \in L^\infty$ implies $\pi^{u,\mathcal{Q}}_\sigma(X^1) \geq \pi^{u,\mathcal{Q}}_\sigma(X^2)$. Finally, a supermartingale property holds: For all stopping times $\sigma \leq \tau \leq T$ and $Q \in \mathcal{Q}$, $\pi^{u,\mathcal{Q}}_\sigma(X) \geq E^Q_\sigma[\pi^{u,\mathcal{Q}}_\tau(X)]$, and $\pi^{u,\mathcal{Q}}_\cdot(X)$ is a supermartingale under any $Q \in \mathcal{Q}$.
b) The sets $\mathcal{M}^e$ and $\mathcal{Q}^{\mathrm{ngd}}$ are m-stable and convex and hence for $\mathcal{Q} = \mathcal{Q}^{\mathrm{ngd}}$, $\pi^u_\cdot(X) = \pi^{u,\mathcal{Q}}_\cdot(X)$ satisfies the properties of Part a).

3.1.1 Parametrizations in an Itô process model

This section describes the Itô process framework for the financial market, and details the parametrizations for dynamic trading strategies and for the no-good-deal constraints. The latter are specified at this stage by abstract correspondences (3.2) such that respective dynamic no-good-deal valuation bounds for contingent claims can be conveniently described in terms of (super-)solutions to BSDEs (Sections 3.1.2-3.1.3) within a convenient framework sufficiently general for all later Sections 3.2-3.3.

We consider models for financial markets where prices $(S^i)_{i=1\dots d}$ of d risky assets evolve according to a stochastic differential equation (SDE)

$$dS_t = \mathrm{diag}(S_t)\sigma_t\left(\xi_t dt + dW_t\right) =: \mathrm{diag}(S_t)\sigma_t d\widehat{W}_t,\ t \in [0,T], \quad S_0 \in (0,\infty)^d,$$

for predictable $\mathbb{R}^d$- and $\mathbb{R}^{d\times n}$-valued coefficients ξ and σ, with $d \leq n$. This includes basically all examples of continuous price and state evolutions in (typically incomplete) markets of the good-deal literature, and permits also for non-Markovian evolutions. Risky asset prices S are given in units of some riskless numéraire asset whose discounted price $S^0 \equiv 1$ is constant. We assume that σ is of maximal rank $d \le n$ (i.e. $\det(\sigma_t \sigma_t^{\mathrm{tr}} \neq 0$, that means no locally redundant assets) and that the market price of risk process ξ, satisfying $\xi_t \in \mathrm{Im}\ \sigma_t^{\mathrm{tr}}$, is bounded. This ensures that market is free of arbitrage but typically incomplete (if $d < n$) in the sense that $\mathcal{M}^e \neq \emptyset$, as the minimal local martingale measure $\widehat{Q}$ given by $d\widehat{Q} = \mathcal{E}\left(-\xi \cdot W\right) dP$ (see [Sch01]) is in $\mathcal{M}^e$, which however is typically not a singleton. Trading strategies are represented by the amount of wealth $\varphi = (\varphi_t^i)_i$ invested in the risky assets $(S^i)_i$. A self-financing trading strategy is described by a pair (V_0, φ), where V_0 is the initial capital while $\varphi = (\varphi_t^i)_i$ describes the amount of wealth invested in the risky assets $(S^i)_i$ at any time t. The set Φ_φ of permitted strategies consists of $\mathbb{R}^d$-valued predictable processes φ satisfying $E^P\left[\int_0^T |\varphi_t^{\mathrm{tr}} \sigma_t|^2 dt\right] < \infty$. For an permitted strategy φ, the associated wealth process V from initial capital V_0 has dynamics $dV_t = \varphi_t^{\mathrm{tr}} \sigma_t d\widehat{W}_t$. To ease notation, we re-parametrize strategies in Φ_φ in terms of integrands $\phi := \sigma^{\mathrm{tr}} \varphi$ with respect to $\widehat{W}$. Indeed, equalities $\phi = \sigma^{\mathrm{tr}} \varphi$ and $\varphi = (\sigma^{\mathrm{tr}})^{-1} \phi$, where $(\sigma^{\mathrm{tr}})^{-1} := (\sigma \sigma^{\mathrm{tr}})^{-1} \sigma$ is the pseudo-inverse of σ^{tr}, provide a one-to-one relation between φ and ϕ. Define the correspondences

$$\Gamma_t(\omega) := \mathrm{Im}\ \sigma_t^{\mathrm{tr}}(\omega) \ \text{ and } \ \Gamma_t^\perp(\omega) := \mathrm{Ker}\ \sigma_t(\omega), \quad (t,\omega) \in [0,T] \times \Omega, \tag{3.5}$$

where $\mathrm{Im}\ \sigma_t^{\mathrm{tr}}$ and $\mathrm{Ker}\ \sigma_t$ denote the range (image) and the kernel of the respective matrices. Clearly, $\mathbb{R}^n = \Gamma_t \oplus \Gamma_t^\perp$ and any $z \in \mathbb{R}^n$ decomposes uniquely into its orthogonal projections as $z = \Pi_{\Gamma_t}(z) \oplus \Pi_{\Gamma_t^\perp}(z) =: \Pi_t(z) \oplus \Pi_t^\perp(z)$. Let

$$\Phi = \Phi_\phi := \left\{ \phi \;\Big|\; \phi \text{ is predictable, } \phi \in \Gamma \text{ and } E\Big[\int_0^T |\phi_t|^2 dt\Big] < \infty \right\}$$

denote the (re-parametrized) set of permitted trading strategies. Proving the claims of the next proposition is routine, using [Roc76] for the first.

Proposition 3.2. *1. The correspondences Γ and $\Gamma^\perp$ are closed-convex-valued and predictable.*

2. $Q \in \mathcal{M}^e$ if and only if $Q \sim P$ with $dQ = \mathcal{E}(\lambda \cdot W)dP$, where λ is predictable and $\lambda = -\xi + \eta$, with $-\xi_t = \Pi_t(\lambda_t) \in \mathrm{Im}\, \sigma_t^{tr}$ and $\eta_t = \Pi_t^\perp(\lambda_t) \in \mathrm{Ker}\, \sigma_t\ \forall t$.

By Part 2 of Proposition 3.2, the set $\mathcal{Q}^{\mathrm{ngd}}$ defined in (3.2) can be written as

$$\mathcal{Q}^{\mathrm{ngd}} = \left\{ Q \sim P \;\Big|\; dQ/dP = \mathcal{E}\left(\lambda \cdot W\right), \lambda \text{ predictable, bounded and } \lambda \in \Lambda \right\}, \tag{3.6}$$

where $\Lambda : [0,T] \times \Omega \rightsquigarrow \mathbb{R}^n$ is defined by $\Lambda_t(\omega) := C_t(\omega) \cap (-\xi_t(\omega) + \mathrm{Ker}\ \sigma)$. By Part 1 of Proposition 3.2 and [Roc76, Corollary 1.K and Theorem 1.M], Λ is a compact-convex-valued predictable correspondence. Slightly beyond the no-free-lunch with vanishing risk condition, we assume that $\mathcal{Q}^{\mathrm{ngd}}$ contains the measure $\widehat{Q}$, or equivalently $-\xi \in C$. This implies that Λ is non-empty valued, hence standard.

3.1.2 Good-deal valuation with uniformly bounded correspondences

We here consider the case where the no-good-deal constraint is described by a uniformly bounded correspondence; a more general case is studied afterwards. We say that a correspondence C is uniformly bounded if it satisfies

Assumption 3.3. $\sup_{(t,\omega)} \sup_{x \in C_t(\omega)} |x| < \infty$.

Let $C : [0,T] \times \Omega \rightsquigarrow \mathbb{R}^n$ be a standard correspondence satisfying Assumption 3.3 and $0 \in C$. Under Assumption 3.3, selections of C are uniformly bounded processes. In particular, the Girsanov kernels of no-good-deal measures are uniformly bounded, and hence boundedness in the definition (3.2) (see also (3.6)) of $\mathcal{Q}^{\mathrm{ngd}}$ is not necessary. The good-deal valuation bound $\pi_t^u(X) := \text{ess sup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} \ E_t^Q[X]$ is well-defined for a contingent claim $X \in L^2 \supset L^\infty$, that may be path-dependent, and one can check that in this case an analog of Part a) of Lemma 3.1 still holds. Though Assumption 3.3 fits well with the classical theory, it would be too restrictive to impose it in general since it may not hold in some interesting practical situations; see for instance the example in Section 3.2.2. Let us recall a fact about linear BSDEs (cf. [EPQ97]) which explains their role for valuation purposes.

Lemma 3.4. *For $Q \sim P$ with bounded Girsanov kernel λ, the linear BSDE*

$$-dY_t = Z_t^{tr} \lambda_t dt - Z_t^{tr} dW_t, \ t \le T, \quad \textit{with} \quad Y_T = X \ \textit{in} \ L^2, \tag{3.7}$$

has a unique standard solution (Y^λ, Z^λ) with $Y_t^\lambda = E_t^Q[X] = Y_0^\lambda + Z \cdot W_t^Q$, and $W^Q := W - \int_0^\cdot \lambda_t dt$. If $X \in L^\infty$ then Y is bounded.

Boundedness of λ in Lemma 3.4 clearly implies that the parameters of the BSDE (3.7) are standard. For unbounded λ, the classical BSDE theory no longer applies and one needs different results to characterize the good-deal bounds in terms of BSDEs. Under Assumption 3.3, Λ is uniformly bounded and thus Girsanov kernels λ^Q for all $Q \in \mathcal{Q}^{\mathrm{ngd}}$ are bounded by the same constant. One has the following

Proposition 3.5. *Let Assumption 3.3 hold.*

1. *For any predictable $\mathbb{R}^n$-valued process Z, there exists a predictable process $\bar\lambda := \bar\lambda(Z) = (\bar\lambda_t(Z_t))_{t \le T} \in \Lambda$ such that $\bar\lambda_t^{tr} Z_t = \underset{\lambda_t \in \Lambda_t}{\text{ess sup}} \ \lambda_t^{tr} Z_t, \ \ t \in [0,T]$.*

2. *For $X \in L^2$, let (Y^λ, Z^λ) (for $\lambda = \lambda^Q \in \Lambda$, $Q \in \mathcal{Q}^{\mathrm{ngd}}$) and (Y, Z) be respectively standard solutions to the classical BSDEs (3.7) and*

$$-dY_t = Z_t^{tr} \bar\lambda_t(Z_t) dt - Z_t^{tr} dW_t, \ t \le T, \ \textit{and} \quad Y_T = X, \tag{3.8}$$

   with $\bar\lambda = \bar\lambda(Z)$ from Part 1. Then $\pi_t^u(X) = \text{ess sup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} \ E_t^Q[X] = E_t^{\bar Q}[X] = Y_t$ holds for $\bar Q \in \mathcal{Q}^{\mathrm{ngd}}$ given by $d\bar Q = \mathcal{E}(\bar\lambda \cdot W) dP$, $Y_t = \text{ess sup}_{\lambda \in \Lambda} \ Y_t^\lambda = Y_t^{\bar\lambda}$.

Proof. Part 1 follows by a direct application of the measurable maximum theorem [Roc76, Theorem 2.K] and measurable selection theorem [Roc76, Theorem 1.C]. As for Part 2, by Assumption 3.3 the parameters of the BSDEs (3.8) and (3.7) are standard. Moreover $\bar{Q}$ is clearly in $\mathcal{Q}^{\mathrm{ngd}}$ since $\bar{\lambda} \in \Lambda$. The remaining of Part 2 hence follows from existence and uniqueness results as well as the comparison theorem for classical BSDEs, cf. [EPQ97, Section 2-3]

□

3.1.3 Good-deal valuation with non-uniformly bounded correspondences

To relax the Assumption 3.3 of uniform boundedness, we now admit for a non-uniformly bounded standard correspondence C, with $0 \in C$, which satisfies

$$\exists\ R \text{ predictable with } \sup_{x \in C_t(\omega)} |x| \leq R_t(\omega)\ \forall (t,\omega) \text{ and } \int_0^T |R_t|^2 dt < \infty. \tag{3.9}$$

It is relevant to look beyond Assumption 3.3, because examples of practical interest require to do so, see Section 3.2.2 where quasi-explicit formulas of good-deal bounds are obtained in a stochastic volatility model, with C not being uniformly bounded but satisfying (3.9). Classical BSDE results do not apply as before to characterize good-deal bounds directly by standard BSDE solutions. Yet, we can still (cf. Theorem 3.7) approximate $\pi^u_\cdot(X)$ for $X \in L^\infty$ by solutions to classical BSDEs for suitable truncations of C, and prove that $\pi^u_t(X)$ coincides with the essential supremum over the larger set $\overline{\mathcal{Q}^{\mathsf{ngd}}} \subseteq \mathcal{M}^e$ given in (3.4). We show, under condition (3.9), that $\pi^u_\cdot(X)$ is the minimal supersolution of the BSDE (3.8). Finally, we show that $\pi^u_\cdot(X)$ is the minimal solution to (3.8) if a worst-case measure $\bar{Q}$ for $\pi^u_0(X)$ exists. Obviously, a maximizing $\bar{Q}$ may be attained rather in the larger set $\overline{\mathcal{Q}^{\mathsf{ngd}}}$.

To this end, let $C^k_t(\omega) = \{x \in C_t(\omega) : |x| \leq k\}$ for $(t,\omega) \in [0,T] \times \Omega$ with $k \in \mathbb{N}$ be a sequence of correspondences. Since C is standard with $0 \in C$, the same holds for each C^k. Clearly, any C^k satisfies Assumption 3.3 and $C^k_t(\omega) \nearrow C_t(\omega)$ as $k \nearrow \infty$. For each $k \in \mathbb{N}$, let $\mathcal{Q}^{\mathrm{ngd}}_k := \mathcal{Q}^{\mathrm{ngd}}_k(S)$ denote the set

$$\mathcal{Q}^{\mathrm{ngd}}_k := \left\{ Q \sim P \;\middle|\; dQ/dP = \mathcal{E}\left(\lambda \cdot W\right),\ \text{with } \lambda \text{ predictable and } \lambda \in \Lambda^k \right\} \tag{3.10}$$

of no-good-deal measures (for S) corresponding to C^k with $\Lambda^k : [0,T] \times \Omega \rightsquigarrow \mathbb{R}^n$ given by $\Lambda^k_t(\omega) := C^k_t(\omega) \cap (-\xi_t(\omega) + \Gamma^\perp_t(\omega))$ and hence also satisfying Assumption 3.3. For $X \in L^2$, we define analogously the bounds $\pi^{u,k}_\cdot(X)$ associated to the sets $\mathcal{Q}^{\mathrm{ngd}}_k$, $k \in \mathbb{N}$ as $\pi^{u,k}_t(X) := \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}_k} E^Q_t[X],\ t \in [0,T]$. The sets $\mathcal{Q}^{\mathrm{ngd}}_k$, $k \in \mathbb{N}$ are m-stable and convex as well.

Lemma 3.6. *(Dynamic principle): Let $\mathcal{Q}$ be a convex and m-stable set of probability measures $Q \sim P$ and $\pi_t^{u,\mathcal{Q}}(X) :=$ ess sup$_{Q\in\mathcal{Q}}$ $E_t^Q[X]$, for $X \in L^\infty$. Then $\pi_\cdot^{u,\mathcal{Q}}(X)$ is the smallest adapted càdlàg process that is a supermartingale under any $Q \in \mathcal{Q}$ with terminal value X.*

Proof. The supermartingale properties of $\pi_\cdot^{u,\mathcal{Q}}(X)$ under every $Q \in \mathcal{Q}$ hold by Part a) of Lemma 3.1. Let Y be another process satisfying the same properties. Then for all $Q \in \mathcal{Q}$ one has $Y_t \geq E_t^Q[X],\ t \in [0,T]$, and taking the essential supremum over $Q \in \mathcal{Q}$ then yields $Y_t \geq \pi_t^{u,\mathcal{Q}}(X)$.

□

Note that since $\mathcal{Q}_k^{\mathrm{ngd}}$, $\mathcal{Q}^{\mathrm{ngd}}$ are convex and m-stable, Lemma 3.6 holds in particular for $\pi_\cdot^u(X) = \pi_\cdot^{u,\mathcal{Q}^{\mathrm{ngd}}}(X)$ and $\pi_\cdot^{u,k}(X) = \pi_\cdot^{u,\mathcal{Q}_k^{\mathrm{ngd}}}(X)$, $k \in \mathbb{N}$. Theorem 3.7 is analogous to Parts 1.-4. of Theorem 2.26 in Chapter 2. We still include its proof in Appendix 3.4, because it seems more instructive in the absence of jumps.

Theorem 3.7. *For any contingent claim $X \in L^\infty$ it holds*

1. $\pi_t^{u,k}(X) \nearrow \operatorname*{ess\,sup}_{Q\in\overline{\mathcal{Q}^{ngd}}} E_t^Q[X] = \pi_t^u(X)$ P-a.s. as $k \nearrow \infty$, for all $t \in [0,T]$.

2. For any $k \in \mathbb{N}$, $\pi_\cdot^{u,k}(X) = Y^k$ for (Y^k, Z^k) being standard solution to the BSDE

$$-dY_t = (\operatorname*{ess\,sup}_{\lambda_t\in\Lambda_t^k} \lambda_t^{tr} Z_t)dt - Z_t^{tr} dW_t,\ t \leq T, \quad \textit{with} \quad Y_T = X. \tag{3.11}$$

3. $\pi_\cdot^u(X)$ and $\pi_\cdot^{u,k}(X)$ for $k \geq \|\xi\|_\infty$ admit Doob-Meyer decompositions

$$\pi_\cdot^u(X) = \pi_0^u(X) + Z \cdot \widehat{W} - A \quad and \quad \pi_\cdot^{u,k}(X) = \pi_0^{u,k}(X) + Z^k \cdot \widehat{W} - A^k, \tag{3.12}$$

under $\widehat{Q}$, where $Z, Z^k \in \mathcal{H}^2(\widehat{Q})$ and A, A^k are non-decreasing predictable processes with $A_T, A_T^k \in L^2(\widehat{Q})$, $A_0^k = A_0 = 0$ and

$$A^k = \int_0^\cdot \Big(\xi_t^{tr} Z_t^k + \operatorname*{ess\,sup}_{\lambda_t\in\Lambda_t^k} \lambda_t^{tr} Z_t^k\Big)dt. \tag{3.13}$$

4. For all $u \leq T$, $A_u^k \to A_u$ weakly in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$ and $Z^k \to Z$ weakly in $L^2(\Omega\times[0,u], \widehat{Q}\otimes dt)$.

Let g be the function defined by

$$g_t(z) := \operatorname*{ess\,sup}_{\lambda_t\in\Lambda_t} \lambda_t^{\mathrm{tr}} z, \quad t \in [0,T],\ z \in \mathbb{R}^n. \tag{3.14}$$

Since g may not be Lipschitz if C does not satisfy Assumption 3.3, then $\pi_\cdot^u(X)$ cannot directly be characterized by classical BSDEs. But one can still obtain a characterization by the minimal supersolution to the BSDE with data (g, X).

Definition 3.8. (Y, Z, K) *is a supersolution of the BSDE with parameters* (f, X) *if*

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t^{tr} dW_t + dK_t \quad \text{for } t \leq T, \ \text{and } Y_T = X,$$

with K *non-decreasing càdlàg adapted, ,* $K_0 = 0$*, and* $\int_0^T |Z_t|^2\, dt < \infty$*. A supersolution with* $K \equiv 0$ *is a BSDE solution. A (super)solution* (Y, Z, K) *is minimal if* $Y_t \leq \bar{Y}_t$*,* $t \in [0, T]$ *holds for any other (super)solution.* $(\bar{Y}, \bar{Z}, \bar{K})$*.*

Note that a minimal supersolution when it exists is unique, as minimality implies uniqueness of the Y-components; since continuous local martingales of finite variation are trivial, identity of the Z- and K-components follows. Existence of the minimal supersolution is sometimes investigated under the condition that there exists at least one supersolution to the BSDE (cf. [DHK13]). This condition is satisfied for the BSDE with parameters (g, X), $X \in L^\infty$ since $g(\cdot, 0) = 0$ and thus $(Y, Z, K) := (|X|_\infty - (|X|_\infty - X)I_{\{T\}}, 0, (|X|_\infty - X)I_{\{T\}})$ is a supersolution. Note that g satisfies $g_t(z) \geq -\xi_t^{\mathrm{tr}} z,\ t \in [0, T]$ and moreover (g, X) satisfies the hypotheses of [DHK13, Theorem 4.17] which implies existence of the minimal supersolution to the BSDE with parameter (g, X). We show that $\pi^u_\cdot(X)$ can be identified with the Y-component of this minimal supersolution. Condition (3.9) ensures that the process $\int_0^\cdot g_t(Z_t)dt$ for g in (3.14) and Z satisfying $\int_0^T |Z_t|^2\, dt < \infty$ is real-valued, since Cauchy-Schwarz inequality would imply $\int_0^T |g_t(Z_t)| dt \leq (\int_0^T |Z_t|^2 dt)^{\frac{1}{2}} (\int_0^T |R_t|^2 dt)^{\frac{1}{2}} < \infty$.

Theorem 3.9. *Let (3.9) hold and* $X \in L^\infty$*. There exists* $Z \in \mathcal{H}^2(\widehat{Q})$ *and a non-decreasing predictable process* K *with* $K_0 = 0$ *such that* $(\pi^u_\cdot(X), Z, K)$ *is the minimal supersolution to the BSDE for data* (g, X) *with* g *from (3.14), and* $\pi^u_\cdot(X) \in \mathcal{S}^\infty$*.*

The proofs for this theorem and for the next corollary are given in Appendix 3.4.

Corollary 3.10. *Let (3.9) hold and* $X \in L^\infty$*. If there exists a measure* $\bar{Q} \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$ *such that* $\pi^u_0(X) = \sup_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E^Q[X] = E^{\bar{Q}}[X]$*, then* $\pi^u_\cdot(X)$ *is a* $\bar{Q}$*-martingale and there exists* $Z \in \mathcal{H}^2(\widehat{Q})$ *such that* $(\pi^u_\cdot(X), Z)$ *is the minimal solution to the BSDE with parameters* (g, X) *for* g *defined in (3.14). The Girsanov kernel* $\bar{\lambda}$ *of* $\bar{Q}$ *satisfies ess sup*$_{\lambda_t \in \Lambda_t} \lambda_t^{tr} Z_t = \bar{\lambda}_t^{tr} Z_t$*, for all* $t \in [0, T]$*.*

For concrete case studies, existence of $\bar{Q} \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$ as in Corollary 3.10 may be shown by direct considerations, see Section 3.2.2 for examples. If one could formulate the no-good-deal restriction so that the set $\overline{\mathcal{Q}^{\mathrm{ngd}}}$ becomes weakly compact in L^1, then $\bar{Q}$ would exist for any $X \in L^\infty$ from maximizing a bounded linear objective functional over a weakly compact subset of L^1. Note that Assumption 3.3 only implies (by Dunford-Pettis compactness theorem [DM78, Chapter II, Theorem 25]) that $\overline{\mathcal{Q}^{\mathrm{ngd}}}$ is weakly relatively compact in L^1. If $\mathcal{Q}^{\mathrm{ngd}}$ is not weakly relatively compact in L^1, then by James' theorem (cf. [AB06, Theorem 6.36]) there exists $X \in L^\infty$ such that the supremum in $\pi^u_0(X) = \sup_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E^Q[X]$ is not attained in the

L^1-closure of $\overline{\mathcal{Q}^{\rm ngd}}$ (since $\overline{\mathcal{Q}^{\rm ngd}}$ is convex) and in particular also not in $\overline{\mathcal{Q}^{\rm ngd}}$. Let us give an example where $\bar{Q}$ does not exist in $\overline{\mathcal{Q}^{\rm ngd}}$ for some contingent claim and C does neither satisfy Assumption 3.3 nor (3.9). Section 3.2.2 will furthermore give an example in a stochastic volatility model where $\bar{Q}$ exists and C is not uniformly bounded but satisfies (3.9).

Example 3.11. *Let $n = 2$ with $W = (W^1, W^2)$, $d = 1$ with $dS_t = S_t\sigma^S dW^1_t$, $S_0 > 0$, $\sigma^S > 0$, and $\xi = 0$. Let $h > 0$ be a deterministic predictable process with $\int_0^T h_t dt = \infty$ and $C_t(\omega) := \{0\} \times [-h_t, h_t]$, $(t,\omega) \in [0,T] \times \Omega$. Now let $X := I_{\{W^2_T \geq 0\}} \in L^\infty$, then $\pi_0 := \sup_{n\in\mathbb{N}} Q^n[\{W^2_T \geq 0\}] \leq \pi^u_0(X) \leq 1$, where $dQ^n = \mathcal{E}(\lambda^n \cdot W^2)dP$ with $\lambda^n_t = h_t \wedge n$, $t \in [0,T]$, $n \in \mathbb{N}$. The process $W^{2,n} := W^2 - \int_0^\cdot \lambda^n_t dt$ is a Q^n-Brownian motion. Hence $W^{2,n}_T \sim \mathcal{N}(0,T)$ under Q^n. We have $\int_0^T \lambda^n_t dt \nearrow \int_0^T h_t dt = \infty$ as $n \nearrow \infty$. Hence $\pi_0 = \sup_{n\in\mathbb{N}} Q^n[\{W^{2,n}_T \geq -\int_0^T \lambda^n_t dt\}] = 1$. Therefore $\pi^u_0(X) = 1$. But there exists no measure $\bar{Q} \in \overline{\mathcal{Q}^{\rm ngd}}$ such that $\pi^u_0(X) = E^{\bar{Q}}_0[X]$. Indeed for such a measure, one would have $\bar{Q}[\{W^2_T \geq 0\}] = 1$ which is not possible since $\bar{Q} \sim P$.*

3.2 Dynamic good-deal hedging

Let again C be a standard correspondence satisfying $0 \in C$, and define the family of a-priori valuation measures

$$\mathcal{P}^{\rm ngd} := \Big\{Q \sim P \;\Big|\; dQ/dP = \mathcal{E}\left(\lambda \cdot W\right),\ \lambda \text{ predictable, bounded, } \lambda \in C\Big\} \tag{3.15}$$

which satisfy the same no-good-deal constraint as those in $\mathcal{Q}^{\rm ngd}$, except that the local martingale condition for S is omitted. One could view $\mathcal{P}^{\rm ngd}$ as the no-good-deal measures for a market only consisting of the riskless asset $S^0 \equiv 1$, i.e. $\mathcal{P}^{\rm ngd} = \mathcal{Q}^{\rm ngd}(1)$. It is natural to define (3.15) as a-priori valuation measures, as the concept of no-good-deal valuation is to consider those risk neutral valuation measures Q, for which any extension of the financial market by additional derivatives price processes (being Q-martingales) would not give rise to 'good deals'; see e.g. [BS06, KS07a, Bec09] for rigorous detail in continuous time for Sharpe ratios, utilities or growth rates; for concepts cf. [Cer03]. Like $\mathcal{Q}^{\rm ngd}$, the set $\mathcal{P}^{\rm ngd}$ clearly is again m-stable and convex. Just as in (3.3), we define the a-priori dynamic coherent risk measure

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\rm ngd}} E^Q_t[X], \quad t \in [0,T], \tag{3.16}$$

for contingent claims $X \in L^2$. Note that $\rho_t(X)$ is well-defined as the essential supremum of finitely valued random variables since measures in $\mathcal{P}^{\rm ngd}$ have bounded Girsanov kernels and hence density processes in $\mathcal{S}^p(P)$ for any $p \in [1,\infty)$.

Elements Q of $\mathcal{P}^{\rm ngd}$ or $\mathcal{Q}^{\rm ngd}$ can be considered as generalized scenarios (as in [ADE$^+$07]). Since $\mathcal{P}^{\rm ngd} \cap \mathcal{M}^e = \mathcal{Q}^{\rm ngd}$ clearly holds, then $\rho_t(X) \geq \pi^u_t(X)$ for all $t \leq T$. An investor holding a liability X and trading in the market according to a permitted trading strategy ϕ, would assign at time t a residual risk $\rho_t(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s)$ to his position. The investor's objective is to hedge his position by a trading strategy $\bar{\phi}$ that minimizes his residual risk at any time $t \leq T$. To justify a premium $\pi^u_\cdot(X)$ for selling X, the minimal capital requirement to make his position ρ-acceptable should coincide with $\pi^u_\cdot(X)$. Thus, his hedging problem is to find a trading strategy $\bar{\phi} \in \Phi$ such that

$$\pi_t^u(X) = \rho_t\Big(X - \int_t^T \bar{\phi}_s^{\mathrm{tr}} d\widehat{W}_s\Big) = \operatorname*{ess\,inf}_{\phi \in \Phi} \ \rho_t\Big(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\Big), \quad t \in [0,T]. \tag{3.17}$$

The good-deal hedging strategy will be defined as a minimizer $\bar{\phi}$ in (3.17), and the good-deal valuation $\pi^u_\cdot(\cdot)$ becomes the market consistent risk measure corresponding to ρ, in the spirit of [BE09]. For a contingent claim X, the tracking error at time $t \in [0,T]$

$$R_t^\phi(X) := \pi_t^u(X) - \pi_0^u(X) - \phi \cdot \widehat{W}_t \tag{3.18}$$

of a hedging strategy $\phi \in \Phi$ is defined as the difference between the dynamic variations in the capital requirement and the profit/loss from trading (hedging) according to ϕ up to time t.

Proposition 3.12. *For $X \in L^2$, let the strategy $\bar{\phi} \in \Phi$ solve (3.17). Then the tracking error $R^{\bar{\phi}}(X)$ is a Q-supermartingale for all $Q \in \mathcal{P}^{ngd}$.*

Proof. By the first equality of (3.17) and the definition of the tracking error it holds $R_t^{\bar{\phi}}(X) = -\pi_0^u(X) + \operatorname{ess\,sup}_{Q \in \mathcal{P}^{\mathrm{ngd}}} \ E_t^Q[X - \int_0^T \bar{\phi}_s^{\mathrm{tr}} d\widehat{W}_s],\ t \in [0,T]$. The claim then follows from m-stability and convexity of $\mathcal{P}^{\mathrm{ngd}}$, applying Lemma 3.1, Part a) extended to $X \in L^2$ by some BSDEs arguments. □

From Remark 2.12 in Chapter 2, let us point out that by Proposition 3.12 we can view the good-deal hedging strategy as being at least mean-self-financing under $Q \in \mathcal{P}^{\mathrm{ngd}}$. The latter is a property that we again interpret as robustness of $\bar{\phi}$ with respect to the set of measure $\mathcal{P}^{\mathrm{ngd}}$ as generalized scenario (in the sense of [ADE$^+$07]). To describe solutions to the hedging problem (3.17), we will often assume that C has further structure and is uniformly bounded. Section 3.2.2 also contains an example for a correspondence C that is not uniformly bounded but satisfies (3.9) in the Heston model, where the hedging problem can be solved in a semi-explicit manner. For a correspondence C satisfying Assumption 3.3, one can describe $\rho_\cdot(X)$ (like $\pi^u_\cdot(X)$ in Proposition 3.5, proof being analogous) by solutions to classical BSDEs:

Proposition 3.13. *Let Assumption 3.3 hold. For $X \in L^2$, let $(\widetilde{Y}, \widetilde{Z})$ and (Y^λ, Z^λ) (for $\lambda \in C$) be the respective standard solutions to the BSDEs*

$$-dY_t = Z_t^{tr} \tilde{\lambda}_t dt - Z_t^{tr} dW_t,\ t \leq T,\ \textit{with } Y_T = X,\ \textit{and} \tag{3.19}$$

$$-dY_t = Z_t^{tr} \lambda_t dt - Z_t^{tr} dW_t,\ t \leq T,\ \textit{with}\ \ Y_T = X, \tag{3.20}$$

where $\tilde{\lambda} = \tilde{\lambda}(Z) \in C$ is a predictable process satisfying the equality $\tilde{\lambda}_t^{tr} Z_t =$ ess sup$_{\lambda_t \in C_t} \lambda_t^{tr} Z_t$ for $t \in [0,T]$. Then the measure $\widetilde{Q}$ with Girsanov kernel $\lambda^{\widetilde{Q}} = \tilde{\lambda}$ is in $\mathcal{P}^{\mathrm{ngd}}$, and $\rho_t(X) = \operatorname*{ess\,sup}_{\lambda \in C} Y_t^\lambda = E_t^{\widetilde{Q}}[X] = \tilde{Y}_t,\ t \in [0,T]$.

3.2.1 Results for ellipsoidal no-good-deal constraints

This section derives more explicit BSDE results to describe the solution to the valuation and the hedging problem (3.17) for (predictable) ellipsoidal no-good-deal constraints. Such generalization includes the important special case of radial constraints (as e.g. in [Bec09]), which is common to the good-deal literature and justified by bounds (uniform in (t,ω)) on optimal growth rates or instantaneous Sharpe ratios, while still permitting comparably explicit results. The generalization could be interpreted as imposing different bounds on growth rates (or Sharpe ratios) for the risk factors associated to the principal axes. While such might appear as rather technical at this stage, in the subsequent context of model uncertainty (cf. Remark 3.24 b)) non-radial constraints will appear naturally.

To this end, let h be a positive bounded predictable process, and A be a predictable $\mathbb{R}^{n\times n}$-matrix-valued process with symmetric values and uniformly elliptic i.e. $A^{\mathrm{tr}} = A$ and $x^{\mathrm{tr}}Ax \geq c\,|x|^2$, for all $x \in \mathbb{R}^n$ and some $c \in \mathbb{R}_+$. The common radial case is achieved by choosing $A \equiv \mathrm{Id}_{\mathbb{R}^n}$. We define the standard (see [Roc76, Corollary 1.Q]) correspondence

$$C_t(\omega) = \Big\{ x \in \mathbb{R}^n \mid x^{\mathrm{tr}} A_t(\omega) x \leq h_t^2(\omega) \Big\}, \quad (t,\omega) \in [0,T] \times \Omega, \tag{3.21}$$

that satisfies Assumption 3.3 due to ellipticity and boundedness of h. Assume that the kernel of the volatility matrix σ is spanned by eigenvectors of A, i.e.

$$A_t^{-1}(\mathrm{Ker}\ \sigma_t) = \mathrm{Ker}\ \sigma_t, \quad t \in [0,T]. \tag{3.22}$$

As the eigenvectors of A are orthogonal and $(\mathrm{Ker}\ \sigma)^\perp = \mathrm{Im}\ \sigma^{\mathrm{tr}}$, then (3.22) can be interpreted as separability of $\mathrm{Im}\ \sigma^{\mathrm{tr}}$ and $\mathrm{Ker}\ \sigma$ in the sense that each of these subspaces has a basis of eigenvectors of A. Given (3.22), the subspaces $\mathrm{Im}\ \sigma^{\mathrm{tr}}$ and $\mathrm{Ker}\ \sigma$ are orthogonal under the scalar product defined by A, one can re-write

$$\mathcal{Q}^{\mathrm{ngd}} = \Big\{ Q \sim P \;\Big|\; dQ/dP = \mathcal{E}(\lambda \cdot W),\ \lambda \text{ predictable},\ \lambda = -\xi + \eta,\ \eta \in C^\xi \cap \mathrm{Ker}\ \sigma \Big\},$$

with $C_t^\xi(\omega) = \{ x \in \mathbb{R}^n \mid x^{\mathrm{tr}} A_t(\omega) x \leq h_t^2(\omega) - \xi_t(\omega)^{\mathrm{tr}} A_t(\omega) \xi_t(\omega) \}$, also satisfying Assumption 3.3. The correspondence C^ξ is standard if

$$h^2 > \xi^{\mathrm{tr}} A \xi, \tag{3.23}$$

The separability condition (3.22) ensures that $-\xi + \eta \in C$ is equivalent to $\eta \in C^\xi$, for $\eta \in \mathrm{Ker}\ \sigma$. This way the ellipsoidal constraint on the Girsanov kernels transfers to one on their

η-component, which permits to formulate the no-good-deal constraint only with respect to non-traded risk factors in the market. In this setup, it is straightforward to obtain an expression for $\bar{\lambda}$ from Part 1 of Proposition 3.5 via

Lemma 3.14. *For $z \in \mathbb{R}^n \setminus \{0\}$, $h > 0$ and a symmetric positive definite $n \times n$-matrix A, the unique maximizer of $y^{tr} z$ subject to $y^{tr} A y \le h^2$ is $\bar{y} = h(z^{tr} A^{-1} z)^{-1/2} A^{-1} z$.*

For $X \in L^2$, since C satisfies Assumption 3.3, there exists a unique standard solution (Y, Z) to the BSDE with terminal condition $Y_T = X$ and

$$dY_t = \left(\xi_t^{\mathrm{tr}} \Pi_t(Z_t) - \sqrt{h_t^2 - \xi_t^{\mathrm{tr}} A_t \xi_t} \sqrt{\Pi_t^{\perp}(Z_t)^{\mathrm{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)} \right) dt + Z_t^{\mathrm{tr}} dW_t. \tag{3.24}$$

We will see that $\pi_\cdot^u(X) = Y$ holds, and that the optimal Girsanov kernel $\bar{\lambda}$ from Part 1 of Proposition 3.5 takes the form $\bar{\lambda} = -\xi + \bar{\eta}$ with $\bar{\eta} \in \mathrm{Ker}\ \sigma$ given by

$$\bar{\eta}_t = \frac{\sqrt{h_t^2 - \xi_t^{\mathrm{tr}} A_t \xi_t}}{\sqrt{\Pi_t^{\perp}(Z_t)^{\mathrm{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)}} A_t^{-1} \Pi_t^{\perp}(Z_t), \quad t \in [0, T]. \tag{3.25}$$

In particular when h tends to $\xi^{\mathrm{tr}} A \xi$ $P \otimes dt$-a.s., then $\bar{\eta}$ tends to 0 and the good-deal bound $\pi_t^u(X)$ converges to $E_t^{\widehat{Q}}[X]$, for $\widehat{Q}$ is the minimal local martingale measure. By Lemma 3.14 and using (3.22), one obtains $\bar{\eta}_t^{\mathrm{tr}} \Pi_t^{\perp}(Z_t) = \text{ess sup}_{\eta_t \in C_t^{\xi} \cap \mathrm{Ker}\ \sigma_t} \eta_t^{\mathrm{tr}} \Pi_t^{\perp}(Z_t)$, and hence $\bar{\lambda}_t^{\mathrm{tr}} Z_t = -\xi_t^{\mathrm{tr}} \Pi_t(Z_t) + \left(h_t^2 - \xi_t^{\mathrm{tr}} A_t \xi_t\right)^{1/2} \left(\Pi_t^{\perp}(Z_t)^{\mathrm{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)\right)^{1/2}$, $t \le T$. Therefore Part 2 of Proposition 3.5 yields

Theorem 3.15. *Assume (3.22) and (3.23) hold. For $X \in L^2$, let (Y, Z) be the standard solution to the BSDE (3.24). Then $\pi_t^u(X) = Y_t = E_t^{\bar{Q}}[X]$, $t \in [0, T]$, where $\bar{Q} \in \mathcal{Q}^{ngd}$ is given by $d\bar{Q} = \mathcal{E}\left((-\xi + \bar{\eta}) \cdot W\right) dP$ with $\bar{\eta}$ given explicitly by (3.25).*

The observation of the following lemma is straightforward.

Lemma 3.16. *The matrices $A_t^{-1}(\omega)$, for $(t, \omega) \in [0, T] \times \Omega$, are positive-definite and satisfy $x^{tr} A_t^{-1}(\omega) x \ge \alpha_t'(\omega) |x|^2$ for all x, t, where $\alpha_t'(\omega) = c \|A_t(\omega)\|^{-2} > 0$ for c being the constant of uniform ellipticity of A. Moreover $\|A\| \ge c$ holds.*

By Lemma 3.14, $\tilde{\lambda} = h(Z^{\mathrm{tr}} A^{-1} Z)^{-1/2}\ A^{-1} Z$ satisfies $\tilde{\lambda}_t^{\mathrm{tr}} Z_t = \text{ess sup}_{\lambda_t^{\mathrm{tr}} A_t \lambda_t \le h_t^2}\ \lambda_t^{\mathrm{tr}} Z_t$, $t \in [0, T]$, with $\tilde{\lambda}_t^{\mathrm{tr}} Z_t = h_t (Z_t^{\mathrm{tr}} A_t^{-1} Z_t)^{1/2}$, $t \in [0, T]$. Hence Proposition 3.13 gives $\rho_t(X) = Y_t$, $t \in [0, T]$, where (Y, Z) uniquely solves the classical BSDE with terminal condition $Y_T = X$ and

$$-dY_t = h_t (Z_t^{\mathrm{tr}} A_t^{-1} Z_t)^{1/2} dt - Z_t^{\mathrm{tr}} dW_t. \tag{3.26}$$

Thanks to Lemma 3.16, a sufficient condition to ensure (3.23) is

$$|\xi| < h\sqrt{\alpha'}. \tag{3.27}$$

In addition it is used to verify for Lemma 3.35 the Kuhn-Tucker conditions before applying the Kuhn-Tucker theorem (see [Roc70, Section 28]), after which comparison results for BSDE yield the result of Theorem 3.17 below. The proof is omitted as it is analogous to that of [Bec09, Theorem 5.4 and Lemma 6.1], using now Lemma 3.35 instead of Lemma 6.1 there. For $\phi \in \Phi$, let (Y^ϕ, Z^ϕ) denote the standard solution to the BSDE with terminal condition $Y_T = X$ and, for $t \le T$,

$$-dY_t = \Big(-\xi_t^{\mathrm{tr}}\phi_t + h_t\big((Z_t-\phi_t)^{\mathrm{tr}}\, A_t^{-1}\, (Z_t-\phi_t) \big)^{1/2} \Big) dt - Z_t^{\mathrm{tr}} dW_t. \tag{3.28}$$

Theorem 3.17. *Assume (3.22),(3.27) hold. For $X \in L^2$, let (Y,Z) and (Y^ϕ, Z^ϕ) (for $\phi \in \Phi$) be standard solutions to the BSDEs (3.24),(3.28). Then $Y_t^\phi = \rho_t(X - \int_t^T \phi_s^{tr} d\widehat{W}_s)$, $t \le T$, and the strategy*

$$\bar{\phi}_t = \frac{\sqrt{\Pi_t^\perp(Z_t)^{tr} A_t^{-1} \Pi_t^\perp(Z_t)}}{\sqrt{h_t^2 - \xi_t^{tr} A_t \xi_t}}\; A_t\xi_t + \Pi_t(Z_t) \tag{3.29}$$

is in Φ and satisfies $Y_t^{\bar{\phi}} = \operatorname*{ess\,inf}_{\phi\in\Phi} Y_t^\phi = Y_t$ for any $t \in [0,T]$, that is

$$\pi_t^u(X) = \operatorname*{ess\,inf}_{\phi\in\Phi} \rho_t\Big(X - \int_t^T \phi_s^{tr} d\widehat{W}_s\Big) = \rho_t\Big(X - \int_t^T \bar{\phi}_s^{tr} d\widehat{W}_s\Big) = Y_t^{\bar{\phi}}.$$

Moreover, the tracking error $R^{\bar{\phi}}(X)$ is a supermartingale under all measures $Q \in \mathcal{P}^{ngd}$ and a martingale under the measure $Q^\lambda \in \mathcal{P}^{ngd}$ with Girsanov kernel

$$\lambda_t := h_t\big((Z_t - \bar{\phi}_t)A_t^{-1}(Z_t - \bar{\phi}_t)\big)^{-1/2} A_t^{-1}\left(Z_t - \bar{\phi}_t\right), \quad t \in [0,T].$$

One could interpret the dynamics of the no-good-deal valuation (3.24) as follows. By $dY_t =: -a_t dt + \Pi_t(Z_t)\xi_t\, dt + Z_t dW_t = -a_t dt + \Pi_t(Z_t) d\widehat{W}_t + \Pi_t^\perp(Z_t) d\widehat{W}$ (cf. Section 3.1.1) it decomposes into a hedgeable part $\Pi_t(Z_t)(\xi_t dt + dW_t) = \Pi_t(Z_t) d\widehat{W}$, that is dynamically spanned by tradeable assets, an orthogonal part $\Pi_t^\perp(Z_t) d\widehat{W}$, being a martingale under P (and $\widehat{Q}$), and a remaining part being an absolutely continuous (finite variation) process whose rate $a_t \ge 0$ may be seen as a premium inherent to the upper good deal bound to compensate the seller of the claim for non-tradeable risk. Note that $a > 0$ on $\{(\omega,t) : \Pi^\perp(Z_t) \ne 0\}$. The summands in the expression (3.29) for the strategy $\bar{\phi}$ play different roles from the perspective of hedging. The second summand is a *non-speculative* component that hedges locally tradeable risk by replication, while the first is a *speculative* component that compensates ("hedges") for unspanned non-tradeable risk by taking favorable bets on the market price of risk. Clearly, good deal bounds fit into the rich theory of g-expectations and market-consistent risk measures (cf. [BE09] and more references therein). See [Lei07] for closely related ideas about instantaneous measurement of risk.

3.2.2 Examples for good-deal valuation and hedging with closed-form solutions

Explicit formulas, if available, facilitate intuition and enable fast computation of valuations, hedges and comparative statics. To this end, several concrete case studies are provided, starting with European options with monotone payoff profiles (e.g. call options) on non-traded assets in a multidimensional model of Black-Scholes type, in which tradeable assets only permit for partial hedging. In parallel to [CT14, Proposition 3, Section 5.3] and [BY08], who employ SDE respectively PDE methods, this demonstrates how previous BSDE analysis can be applied in concrete case studies and we contribute some slight generalizations as well (e.g. higher dimensions, ellipsoidal constraints). As a further example, we contribute new explicit formulas for an option to exchange (geometric averages of) non-traded assets into traded assets. As before, the no-good-deal approach here gives rise to a familiar option pricing formula (by Margrabe) but suitable adjustments of parameter inputs are required, showing the difference to a simple no-arbitrage valuation approach that uses only one (given) single risk neutral measure. A further example derives semi-explicit good-deal solutions for the stochastic volatility model by Heston, for no-good-deal constraints on market prices of (unspanned) stochastic volatility risk which impose an interval range on the mean reversion level of the stochastic variance process under any valuation measure $Q \in Q^{\mathrm{ngd}}$. Technically, this corresponds to imposing bounds on the instantaneous Sharpe ratio which are inversely proportional to the stochastic volatility. This is different to a related result by [BL09], in that their example imposes no good deal constraints in terms of bounds on simultaneous changes in the level of mean-reversion combined with opposite changes in reversion speed. We emphasize that, in addition to valuation formulas, all our examples provide explicit results for good-deal hedging strategies as well. Detailed derivations of the formulas in Sections 3.2.2-3.2.2 are given in Appendix 3.4

Closed-form formulas for options in a generalized Black-Scholes model

The market information $\mathbb{F} = (\mathcal{F}_t)_{t\le T}$ is generated by an n-dimensional P-Brownian motion $W := (W^1, \dots, W^n)^{\mathrm{tr}}$ with $W^S = (W^1, \dots, W^d)^{\mathrm{tr}}$, $d < n$ for $n, d \in \mathbb{N}$, and is augmented by null-sets. The financial market consists of $d \le n$ (incomplete if $d < n$) stocks with (discounted) prices $S = (S^k)_{k=1}^d$ and further $n-d$ non-traded assets with values $H = (H^l)_{l=1}^{n-d}$. We consider a risk neutral model ($P = \widehat{Q} \in \mathcal{M}^e,\ \xi = 0$) where the processes S and H evolve as

$$dS_t = \mathrm{diag}(S_t)\sigma^S dW_t^S \quad \text{and} \quad dH_t = \mathrm{diag}(H_t)\big(\gamma dt + \beta dW_t\big), \quad t \in [0,T],$$

with $S_0 \in (0,\infty)^d,\ H_0 \in (0,\infty)^{n-d}$, constant coefficients $\sigma^S = (\sigma^S_{ki})_{k,i} \in \mathbb{R}^{d\times d}$ invertible, $\beta = (\beta_{li})_{l,i} \in \mathbb{R}^{(n-d)\times n}$ and $\gamma \in \mathbb{R}^{n-d}$. The volatility matrix of S is $\sigma := (\sigma^S, 0) \in \mathbb{R}^{d\times n}$ and is clearly of maximal rank $d \le n$. For $z \in \mathbb{R}^n$, we have $\Pi(z) = (z^1, \dots, z^d, 0, \dots, 0)^{\mathrm{tr}} \in \mathbb{R}^n$ and $\Pi^\perp(z) = \Big(0, \dots, 0, z^{d+1}, \dots, z^n\Big)^{\mathrm{tr}} \in \mathbb{R}^n$. We assume the ellipsoidal framework of

Section 3.2.1, with $h \equiv \mathrm{const} > 0$ and $A \equiv \mathrm{diag}(a)$, with $a \in (0,\infty)^n$. Clearly A satisfies the assumption (3.22). By convention, set $0/0 = 0$. From Theorem 3.15 we know that $\pi_t^u(X) = Y_t = E_t^{\bar{Q}}[X]$ with $d\bar{Q}/dP = \mathcal{E}\left(\bar{\lambda}\cdot W\right)$ where

$$\bar{\lambda}_t = h\big(\sum_{i=d+1}^{n}(Z_t^i)^2/a_i\big)^{-1/2}(0,\ldots,0,Z_t^{d+1}/a_{d+1},\ldots,Z_t^n/a_n)^{\mathrm{tr}},$$

for (Y,Z) solving the classical BSDE

$$-dY_t = h\big(\sum_{i=d+1}^{n}(Z_t^i)^2/a_i\big)^{1/2}dt - Z_t^{\mathrm{tr}}dW_t,\ t \le T, \quad \text{and} \quad Y_T = X. \tag{3.30}$$

By Theorem 3.17 the good-deal hedging strategy is $\bar{\phi}_t = \Pi(Z_t),\quad t \le T$. Define the geometric averages $\tilde{S}_t = \big(\prod_{k=1}^{d} S_t^k\big)^{1/d}$ and $\tilde{H}_t = \big(\prod_{l=1}^{n-d} H_t^l\big)^{1/(n-d)}$. Then one can rewrite $\tilde{S}_t = \tilde{S}_0\exp\left(\tilde{\sigma}^{\mathrm{tr}}W_t^S + (\tilde{\mu} - \frac{1}{2}|\tilde{\sigma}|^2)t\right)$ and $\tilde{H}_t = \tilde{H}_0\exp\left(\tilde{\beta}^{\mathrm{tr}}W_t + (\tilde{\gamma} - \frac{1}{2}|\tilde{\beta}|^2)t\right)$, where $\tilde{\sigma} := \frac{1}{d}(\sigma^S)^{\mathrm{tr}}\mathbb{1}$, $\tilde{\mu} := \frac{1}{2}|\tilde{\sigma}|^2 - \frac{1}{2d}|\sigma^S|^2$, $\tilde{\beta} = \frac{1}{n-d}\beta^{\mathrm{tr}}\mathbb{1}$ and $\tilde{\gamma} := \frac{1}{n-d}\gamma^{\mathrm{tr}}\mathbb{1} + \frac{1}{2}|\tilde{\beta}|^2 - \frac{1}{2(n-d)}|\beta|^2$, with $\mathbb{1} = (1,\ldots,1)^{\mathrm{tr}}$. We treat the following two examples.

European option on non-traded assets: Consider a European option $X = G(\tilde{H}_T)$ in L^2 on the geometric average $\tilde{H}$, where $x \mapsto G(x)$ is a non-decreasing measurable pay-off function of polynomial growth in $x^{\pm 1}$, i.e. $|G(x)| \le k(1 + x^n + x^{-n})$ for all $x > 0$, for some $k > 0$ and $n \in \mathbb{N}$. We claim that $\bar{\lambda}$ is a constant process given by $\bar{\lambda}_t = h\big(\sum_{i=d+1}^{n}\tilde{\beta}_i^2/a_i\big)^{-1/2}(0,\ldots,0,\tilde{\beta}_{d+1}/a_{d+1},\ldots,\tilde{\beta}_n/a_n)^{\mathrm{tr}}$. Let (Y,Z) be the standard solution to the linear classical BSDE (3.7) with $\lambda = \bar{\lambda}$. For $\bar{\lambda}$ constant, the Feynman-Kac formula yields $Y_t = E_t^{\bar{Q}}[G(\tilde{H}_T)] = u(t,\tilde{H}_t)$ and $Z_t = \tilde{H}_t\partial_x u(t,\tilde{H}_t)\tilde{\beta}$ for a function $u \in \mathcal{C}^{1,2}([0,T)\times(0,\infty))$ solution to a Black-Scholes type PDE (after coordinate transformations that reduce the PDE into the heat equation using [KS06, Section 4.3]). Since G is non-decreasing, $\partial_x u \ge 0$ holds. Because of this one actually has $\bar{\lambda}_t^{\mathrm{tr}}Z_t = h\big(\sum_{i=d+1}^{n}(Z_t^i)^2/a_i\big)^{1/2},\ t \in [0,T]$. Hence $\bar{\lambda}$ is indeed the constant process given above, and $\pi_t^u(X) = E_t^{\bar{Q}}[G(\tilde{H}_T)]$. The process $\tilde{H}$ satisfies $\tilde{H}_t = \tilde{H}_0\ e^{\alpha_+ t}\exp\big(\tilde{\beta}^{\mathrm{tr}}\bar{W}_t - \frac{1}{2}|\tilde{\beta}|^2 t\big)$, $t \in [0,T]$, where $\alpha_\pm := \tilde{\gamma} \pm h\big(\sum_{i=d+1}^{n}\tilde{\beta}_i^2/a_i\big)^{1/2}$ and $\bar{W}$ is an n-dimensional $\bar{Q}$-Brownian motion. Specifically for $G(x) := (x-K)^+$, X is a call option on $\tilde{H}$ with strike K and maturity T. The upper good-deal bound is given for $t \in [0,T]$ by a Black-Scholes type formula (with "vol" abbreviating volatility)

$$\begin{aligned}\pi_t^u(X) &= N(d_+)\tilde{H}_t e^{\alpha_+(T-t)} - KN(d_-)\\ &= e^{\alpha_+(T-t)} * \text{B/S-call-price(time: } t, \text{ spot: } \tilde{H}_t, \text{ strike: } Ke^{-\alpha_+(T-t)}, \text{vol: } |\tilde{\beta}|),\end{aligned} \tag{3.31}$$

where $d_\pm := \big(\ln(\tilde{H}_t/K) + (\alpha_+ \pm \frac{1}{2}|\tilde{\beta}|^2)(T-t)\big)\big(|\tilde{\beta}|\sqrt{T-t}\big)^{-1}$ and N is the cdf of the standard normal law. Analogously, the lower good-deal bound turns out as

$$\pi_t^l(X) = e^{\alpha_-(T-t)} * \text{B/S-call-price(time: } t, \text{ spot: } \tilde{H}_t, \text{ strike: } Ke^{-\alpha_-(T-t)}, \text{vol: } |\tilde{\beta}|).$$

The difference between the good-deal valuation formulas above and the standard Black-Scholes formula for risk-neutral valuation $E_t^{\widehat{Q}}[X]$ under measure $P = \widehat{Q}$ for a call option $(\tilde{H}_T - K)^+$ shows in the factors $e^{\alpha_\pm(T-t)}$ multiplying the spot price $\tilde{H}_t$, which reduce to the risk-neutral case $e^{\tilde{\gamma}(T-t)}$ if $h = 0$, i.e. $\alpha_\pm = \tilde{\gamma}$. The difference $\alpha_\pm - \tilde{\gamma} = \pm h(\sum_{i=d+1}^n \tilde{\beta}_i^2/a_i)^{1/2}$ when $h > 0$ translates into an additional premium an option trader (selling at $\pi_\cdot^u(X)$ or buying at $\pi_\cdot^l(X)$) would require, if using the no-good-deal approach instead of the arbitrage-free valuation under a given risk neutral measure $P = \widehat{Q}$ (being an element of $\mathcal{Q}^{\text{ngd}}$). The good-deal hedging strategy for the seller of X in terms of parametrizations of Section 3.1.1 is

$$\bar{\phi}_t = e^{\alpha_+(T-t)} N(d_+) \tilde{H}_t (\tilde{\beta}_1, \ldots, \tilde{\beta}_d, 0, \ldots, 0)^{\text{tr}}, \ t \in [0, T], \tag{3.32}$$

which coincides with the delta hedging strategy (as computed under $P = \widehat{Q}$) for the call option $(\tilde{H}_T - K)^+$ only if α_+ is zero and the risky asset $\tilde{H}$ is tradeable. The hedging strategy of the buyer is derived analogously.

Exchange option of traded and non-traded assets: Consider an European option to exchange the traded asset $\tilde{S}$ for the non-traded asset $\tilde{H}$ at maturity T with payoff $X = (\tilde{H}_T - \tilde{S}_T)^+ \in L^2$. The upper bound $\pi_t^u(X) = E_t^{\bar{Q}}[X]$ can be explicitly derived (see Appendix) using arguments from the previous example in combination with a change of numéraire. We thereby obtain a Margrabe type formula

$$\begin{aligned} \pi_t^u(X) &= N(d_+) \tilde{H}_t e^{\alpha_+(T-t)} - N(d_-) \tilde{S}_t e^{\tilde{\mu}(T-t)} \\ &= \text{B/S-call-price(time: } t, \text{ spot: } \tilde{H}_t e^{\alpha_+(T-t)}, \text{ strike: } \tilde{S}_t e^{\tilde{\mu}(T-t)}, \text{ vol: } \delta), \end{aligned} \tag{3.33}$$

where $d_\pm := \big(\ln(\tilde{H}_t/\tilde{S}_t) + (\alpha_+ + \tilde{\mu} \pm \frac{\delta^2}{2})(T-t)\big)\big(\delta\sqrt{T-t}\big)^{-1}$. Analogously, the corresponding lower good-deal bound is

$$\pi_t^l(X) = \text{B/S-call-price(time: } t, \text{ spot: } \tilde{H}_t e^{\alpha_-(T-t)}, \text{ strike: } \tilde{S}_t e^{\tilde{\mu}(T-t)}, \text{ vol: } \delta).$$

The good-deal hedging strategy $\bar{\phi}_t$ for the seller of the exchange option equals

$$N(d_+) \tilde{H}_t e^{\alpha_+(T-t)} (\tilde{\beta}_1, \ldots, \tilde{\beta}_d, 0, \ldots, 0)^{\text{tr}} - N(d_-) \tilde{S}_t e^{\tilde{\mu}(T-t)} (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_d, 0, \ldots, 0)^{\text{tr}}.$$

Again, the difference between the good-deal valuation formula and the classical Margrabe formula, as computed by standard no-arbitrage valuation under risk neutral measure $P = \widehat{Q}$, for the exchange option $(\tilde{H}_T - \tilde{S}_T)^+$ shows by the presence of the factors $e^{\alpha_\pm(T-t)}$ involving the term $\pm h(\sum_{i=d+1}^n \tilde{\beta}_i^2/a_i)^{1/2}$, which depends only on the parameters A and h for no-good-deal restrictions.

Computational results by Monte Carlo

To demonstrate that good-deal bounds and hedging strategies can be computed numerically in moderately high dimensions by generic simulation methods available for classical BSDE, we apply the (generic) multilevel Monte Carlo algorithm from [BT14] (that builds on [GT15]) to approximate the solution (Y, Z) of the BSDE (3.30) in dimension $n = 4$, and compare with the known analytical solution for the exchange option $X := (\tilde{H}_T - \tilde{S}_T)^+$. Using parameters $d = 2$, $T = 1$ and

$$H_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma^S = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.4 \end{pmatrix}, \quad \gamma = (0.1,\ 0.3)^{\mathsf{tr}},$$
$$\beta = \begin{pmatrix} 0.3 & 0.4 & 0.2 & 0.5 \\ 0.5 & 0.7 & 0.3 & 0.4 \end{pmatrix}, \quad h = 0.3, \quad \text{and} \quad A = \mathrm{diag}(0.5,\ 0.65,\ 0.8,\ 0.95),$$

we compare the approximate values at time $t = 0$ to the known theoretical values obtained from Section 3.2.2. The exact value of the good-deal bound at time $t = 0$ according to the formula (3.33) is then $\pi_0^u(X) = 0.5494$, up to four digits, while for the hedging strategy it is $\bar{\phi}_0 = (0.3049, 0.4440, 0, 0)$, the exact value of Z_0 being $(0.3049, 0.4440, 0.2792, 0.5025)$. We use a 4-level algorithm on an equidistant time grid with $N = 2^4$ steps, a number of sample paths $M = 3 \times 10^6$ and with $K = 50^4$ regression functions, being indicator functions on a hypercube partition of $\mathbb{R}^4$, the state space of the forward process (S, H). Table 3.1 provides the numerical simulation results, summarized by the approximation means for the good-deal bound and the hedging strategy at time 0, the empirical root-mean-square errors (RMSE) computed coordinate-wise and the corresponding relative values (Rel.RMSE), based on 80 independent simulation runs. Simulation in Matlab for one run took 153sec on a core-i7 cpu laptop, showing relative errors (in terms of maximal coordinates in Rel.RMSE) of about 0.07% for valuation and 0.34% for hedging.

	Y_0 approx	Z_0 approx	$\bar{\phi}_0$ approx
Mean	0.5499	$(0.3052, 0.4462, 0.2852, 0.5137)$	$(0.3052, 0.4462, 0, 0)$
RMSE	$10^{-4} \times 4$	$10^{-4} \times (10, 13, 12, 13)$	$10^{-4} \times (10, 13, 0, 0)$
Rel.RMSE	$10^{-4} \times 7$	$10^{-4} \times (34, 29, 41, 27)$	$10^{-4} \times (34, 29, 0, 0)$

Table 3.1: Mean and (relative) root-mean-square errors of approximations

Semi-explicit formulas in the Heston stochastic volatility model

The market information is generated by a 2-dimensional P-Brownian motion $W = (W^S, W^\nu)$, and is augmented by null-sets. We are going to consider a European put option $X = (K - S_T)^+$ on S with strike K in the Heston model

$$dS_t = S_t\sqrt{\nu_t}\, dW_t^S \quad \text{and} \quad d\nu_t = b(\frac{a}{b} - \nu_t)dt + \beta\sqrt{\nu_t}(\rho dW_t^S + \sqrt{1-\rho^2}dW_t^\nu),\ t \le T,$$

that is specified directly under a risk neutral measure $P = \widehat{Q}$, with $S_0, \nu_0 > 0$, $a, b, \beta > 0$ and $\rho \in (-1, 1)$. Here the variance process ν is a CIR process with b representing the mean-reversion speed, a/b the mean-reversion level and $\beta/2$ the volatility of the variance. Assume that the condition $\beta^2 \le 2a$ is satisfied, such that by the Feller's test for explosions (cf. [KS06, Theorem 5.5.29]) applied to the process $\ln(\nu)$ the variance process ν is strictly positive. In the sequel we refer to this condition (i.e. $\beta^2 \le 2a$) for a CIR process as the Feller condition. The equivalent local martingale measures $Q \in \mathcal{M}^e$ in this model are specified by Girsanov kernels λ such that $dQ/dP = \mathcal{E}(\lambda \cdot W^\nu)$ is a uniformly integrable martingale. Indeed, we parametrize the pricing measures only by the second component of their Girsanov kernels (i.e. with respect to W^ν) since the first component is always zero. We consider the no-good-deal constraint correspondence

$$C_t(\omega) = \{x \in \mathbb{R}^2 : |x| \le \varepsilon/\sqrt{\nu_t(\omega)}\} \quad (t,\omega) \in [0,T] \times \Omega, \tag{3.34}$$

for a constant $\varepsilon > 0$. One observes that C is standard with $0 \in C$, non-uniformly bounded and satisfies (3.9) for $R = \varepsilon/\sqrt{\nu}$ (since $\nu > 0$ is continuous). Hence good-deal valuation results for uniformly bounded correspondences may not apply. Using [CFY05], we can obtain a convenient Heston-type formula (semi-explicit, computation requiring only 1-dim. integration) for the good-deal bound of the put option $X = (K - S_T)^+$,

$$\pi_t^u(X) = \text{ Heston-put-price(time: } t, \bar{a} := a + \beta\varepsilon\sqrt{1-\rho^2}, b, \beta), \tag{3.35}$$

just like the ordinary Heston put price, associated to parameters (t, a, b, β), but where the parameter a has to be adjusted to $\bar{a} := a + \beta\varepsilon\sqrt{1-\rho^2}$. The formula for the lower bound $\pi_t^l(X)$ is similar, but with $\bar{a}$ replaced by $\underline{a} := a - \beta\varepsilon\sqrt{1-\rho^2}$, for which the Feller condition $\beta^2 \le 2\underline{a}$ is still satisfied if $\varepsilon \le \frac{1}{2}\beta^{-1}(2a - \beta^2)(1-\rho^2)^{-1/2}$. In particular, $\pi_t^u(X) = E_t^{\bar{Q}}[X]$ holds with $d\bar{Q}/dP = \mathcal{E}((\varepsilon/\sqrt{\nu}) \cdot W^\nu)$. By Corollary 3.10 this yields $\bar{Y} = \pi_\cdot^u(X)$ for the minimal solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty \times \mathcal{H}^2$ of the BSDE

$$-dY_t = \frac{\varepsilon}{\sqrt{\nu_t}}|Z_t^2| - Z_t^{\mathrm{tr}} dW_t,\ t \in [0,T], \quad Y_T = (K - S_T)^+. \tag{3.36}$$

The (seller's) good-deal hedging strategy $\bar{\phi}$ is given by the semi-explicit formula

$$\bar{\phi}_t = S_t\sqrt{\nu_t}\ \Delta_t + \frac{\beta\rho}{2}\ \mathcal{V}_t, \tag{3.37}$$

where Δ_t and $\mathcal{V}_t$ denote the delta and the vega of the put option at time t in the Heston model with parameters $(\bar{a}, b, \beta)$. Derivations are provided in Appendix 3.4. We note that (3.37) coincides (cf. [PSHE09]) with the risk-minimizing strategy (in the sense of [Sch01]) for the put in a Heston model, not with respect to the probability P but with respect to the measure $\bar{Q}$ (derived just before) under which also Heston dynamics but with modified parameters prevail. This shows, how the strategy (3.37) differs from the standard risk minimizing strategy under P (as in [PSHE09, HPS01]). Good-deal valuation bounds for a put option in the Heston model are thus given by a Heston type formula but for a mean-reversion level increased by $\beta\varepsilon\sqrt{1-\rho^2}/b > 0$. Similar to earlier examples, this difference constitutes an increase in the premium that an issuer selling at $\pi^u_\cdot(X)$ would require according to good-deal valuation, in comparison to a standard arbitrage free valuation under one given risk neutral measure $P = \widehat{Q}$, when S is the only risky asset available for hedging and stochastic volatility risk is otherwise taken to be unspanned.

Figures 3.1,3.2,3.3 graphically illustrate this, showing the good-deal valuations $\pi^u_0(X), \pi^l_0(X)$ (at $t = 0$) for a long-dated put option with maturity $T = 10$ in relation to the underlying S_0, to the correlation coefficient ρ and to the no-good-deal constraint parameter ε (for bound on optimal growth rate $h = \varepsilon/\sqrt{\nu}$) respectively. Other global parameters are $K = 100,\ a = 0.12,\ b = 3,\ \beta = 0.3,\ \nu_0 = 0.04$. Computations of the Heston formula have been done in Matlab following the algorithm of [KJ05]. Figure 3.1 is a plot of $\pi^u_0(X), \pi^l_0(X)$ as function of initial stock price S_0 for values of ε in $\{0.15, 0.25\}$. Similarly Figure 3.2 provides a plot illustrating the variation with ρ for $\varepsilon \in \{0.1, 0.2\}$, while Figure 3.3 illustrates the dependence on ε. The largest value 0.35 for ε in Figure 3.3 has been chosen as the maximal one allowing for the Feller condition $\beta^2 \le 2\underline{a} = a - \beta\varepsilon\sqrt{1-\rho^2}$ for the lower bound $\pi^l_0(X)$ to be satisfied, i.e. $\varepsilon \le \frac{1}{2}\beta^{-1}(2a - \beta^2)(1-\rho^2)^{-1/2} \approx 0.35$ for the chosen parameters. Because the values of ε are close to zero, the lines in Figure 3.3 may look straight at the first impression, but by having a closer look the reader can convince himself that the lines are indeed not straight as expected. We could have plotted the upper bound $\pi^u_0(X)$ for larger values of ε, but we simply chose to use on the same plot the same range of ε as that for the lower bound $\pi^l_0(X)$. The standard Heston price computed directly under a given risk neutral (minimal martingale) measure $P = \widehat{Q}$ (i.e. for $\varepsilon = 0$) lies between the upper and lower good-deal bounds, whose spread increases with $\varepsilon > 0$. The monotonicity in ε is intuitively obvious since as ϵ increases, the correspondence C maps to larger sets, yielding weaker no-good-deal constraints which then imply wider good-deal valuation bounds. That the bounds in Figure 3.2 coincide for perfect correlation $\rho \in \{-1, 1\}$ is also intuitively clear. Indeed since for $|\rho| = 1$ volatility risk is entirely spanned by the tradeable asset, then the former can be perfectly hedged such that the Heston model becomes complete and $\pi^u_0(X) = \pi^l_0(X) = E^{\widehat{Q}}[X]$ holds for all contingent claims X.

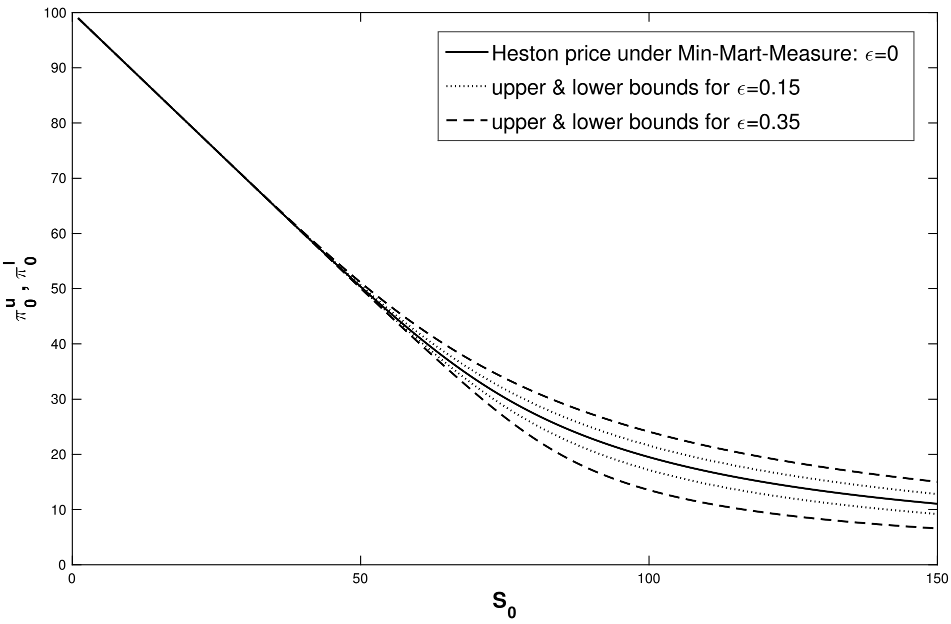


Figure 3.1: Dependence of $\pi_0^u(X), \pi_0^l(X)$ on S_0 for $\rho = -0.7$ and $T = 10$.

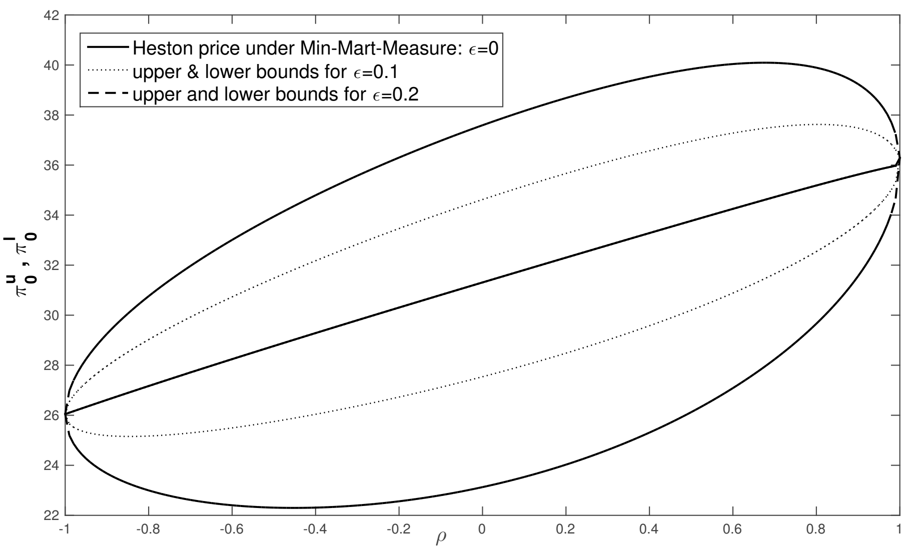


Figure 3.2: Dependence of $\pi_0^u(X), \pi_0^l(X)$ on ρ for $S_0 = 100$ and $T = 20$.

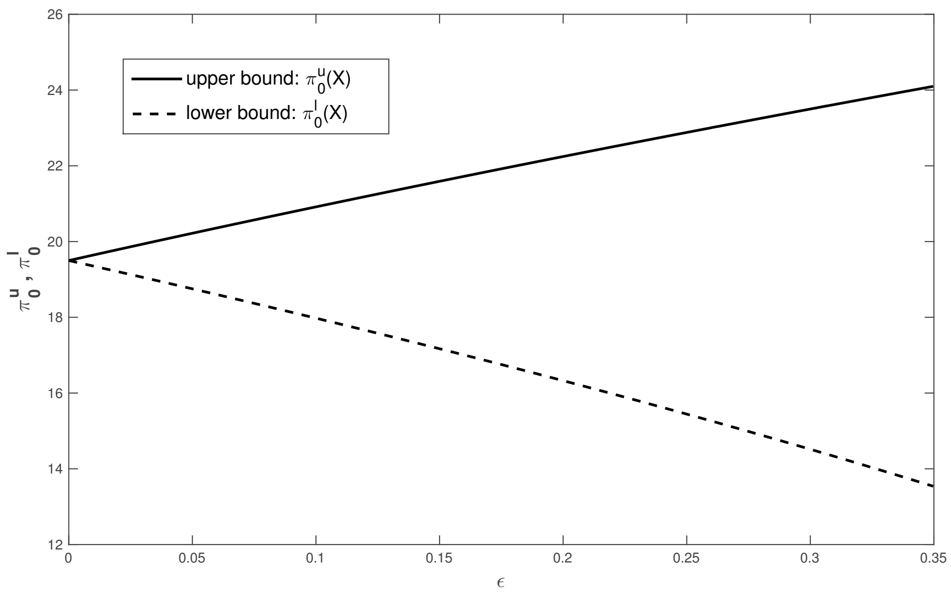


Figure 3.3: Dependence of $\pi_0^u(X), \pi_0^l(X)$ on ε for $S_0 = 100$, $\rho = -0.7$ and $T = 10$.

3.3 Good-deal valuation and hedging under model uncertainty

In preceding sections, good-deal bounds and hedging strategies have been described by classical BSDEs under the probability measure P, expressing the objects of interest in terms of the market price of risk ξ with respect to P. In reality, the objective real world probability measure is not precisely known, hence there is ambiguity about the market price of risk. To include model uncertainty (ambiguity) into the analysis, we follow a multiple priors approach in spirit of [GS89, CE02, ES03], by specifying a confidence region of reference probability measures $\{P^\theta : \theta \in \Theta\}$ (multiple priors, interpreted as potential real world probabilities of equal right), centered around some measure P_0. In practice, an investor facing model uncertainty may first extract an estimate P_0 for the true but uncertain P from data, but then consider a class $\mathcal{R}$ of potential reference measures in some confidence region around P_0 to acknowledge the statistical uncertainty of estimation. Starting point for good-deal valuation approach under uncertainty is then to associate to each model P^θ its own family of (a-priori) no-good-deal measures $\mathcal{Q}^{\mathrm{ngd}}(P^\theta)$ (resp. $\mathcal{P}^{\mathrm{ngd}}(P^\theta)$). A robust worst-case approach requires the seller of a derivative to consider the (worst-case) model $P^{\bar{\theta}}$ that provides the largest upper good-deal valuation bound, to be conservative against model misspecification (see (3.58)). Such leads to wider good-deal bounds, corresponding to a larger overall set of no-good-deal measures under uncertainty. Notably, it will simultaneously also give rise to a suitable robust notion of good-deal hedging, which is uniform with respect to all P^θ, by means of a saddle point result that ensures a minmax identity (see Theorem 3.30). We associate to each model P^θ a correspondence C^θ that defines the set of no-good-deal measures in this model. The aggregate set of no-good-deal measures will be described then by single correspondence $\widetilde{C}$, which incorporates also the uncertainty. Technically, this makes is possible to apply analysis obtained in the framework of previous sections of this chapter, with P_0 taking the role of P.

3.3.1 Model uncertainty framework

Let $(\Omega, \mathcal{F}, P_0, \mathbb{F})$ be a probability space with a usual filtration $\mathbb{F} = (\mathcal{F}_t)_{t\le T}$ generated by an n-dimensional P_0-Brownian motion W^0. We assume that all reference measures P^θ are equivalent to P_0 with corresponding Girsanov kernels θ evolving in some given confidence region Θ. More precisely, we define

$$\mathcal{R} := \Big\{ P^\theta \sim P_0 \;\Big|\; dP^\theta/dP_0 = \; \mathcal{E}\left(\theta \cdot W^0\right), \text{ with } \theta \text{ predictable and } \theta \in \Theta \Big\},$$

where $\Theta : [0,T] \times \Omega \rightsquigarrow \mathbb{R}^n$ is a standard correspondence satisfying Assumption 3.3 and $0 \in \Theta$, hence $P_0 \in \mathcal{R} \neq \emptyset$. A similar framework has been considered for example in [CE02, Que04] for solving the robust utility maximization problem under Knightian uncertainty about drift coefficients. We do write $\theta \in \Theta$ for θ being a predictable selection of Θ.

The financial market consists of $d \le n$ tradeable risky assets whose discounted prices $(S^i)_{i=1}^d$ under P^θ (for $\theta \in \Theta$) evolve as Itô processes, solving the SDEs

$$dS_t = \mathrm{diag}(S_t)\sigma_t(\xi_t^\theta dt + dW_t^\theta) =: \mathrm{diag}(S_t)\sigma_t d\widehat{W}_t^\theta, \quad t \le T, \tag{3.38}$$

with $S_0 \in (0,\infty)^d$, for $\mathbb{R}^n$-valued predictable ξ^θ and $\mathbb{R}^{d\times n}$-valued predictable volatility σ of full rank, and $W^\theta := W^0 - \int_0^\cdot \theta_s ds$ a P^θ-Brownian motion. Noting that market prices of risk, ξ_t^θ and ξ_t^0, canonically take values in $\mathrm{Im}\ \sigma_t^{\mathrm{tr}}$, we assume that market prices of risk ξ^θ (under P^θ for $\theta \in \Theta$) have the form

$$\xi_t^\theta = \xi_t^0 + \Pi_t(\theta_t) \in \mathrm{Im}\ \sigma_t^{\mathrm{tr}},\ t \in [0,T], \tag{3.39}$$

and that ξ^0 is bounded. By (3.39), the solutions of the SDEs (3.38) coincide P_0-a.s. for all $\theta \in \Theta$. The process ξ^θ (for $\theta \in \Theta$) is the market price of risk in the model P^θ and is also bounded (since ξ^0 is bounded and Θ satisfies Assumption 3.3). Hence, the minimal martingale measure [Sch01] $\widehat{Q}^\theta$ with respect to P^θ is $d\widehat{Q}^\theta = \mathcal{E}(-\xi^\theta \cdot W^\theta)dP^\theta$. In addition $d\widehat{Q}^\theta = \mathcal{E}\left(\Pi^\perp(\theta)\cdot \widehat{W}^0\right) d\widehat{Q}^0$ and $\widehat{W}^\theta = \widehat{W}^0 - \int_0^\cdot \Pi_t^\perp(\theta_t)dt$, for all $\theta \in \Theta$. We recall from Section 3.1.1 how dynamic trading strategies are defined and re-parametrized in terms of integrands $(\phi^i)_{i=1}^d$ with respect to $\widehat{W}^0$. The set of permitted trading strategies is

$$\Phi := \Big\{ \phi \;\Big|\; \phi \text{ is predictable, } \phi \in \mathrm{Im}\ \sigma^{\mathrm{tr}} \text{ and } E^{P_0}\Big[\int_0^T |\phi_t|^2 dt\Big] < \infty \Big\}.$$

Since $\phi^{\mathrm{tr}}\Pi^{\perp}(\theta) = 0$ for $\theta \in \Theta$, the wealth process V^{ϕ} of strategy $\phi \in \Phi$ with initial capital V_0 is $V^{\phi} = V_0 + \phi \cdot \widehat{W}^{\theta} = V_0 + \phi \cdot \widehat{W}^{0}$, for all $\theta \in \Theta$. Let $\mathcal{M}^e(P^{\theta}) := \mathcal{M}^e(S, P^{\theta})$ denote the set of equivalent local martingale measures for S in the model P^{θ}. Noting $P^{\theta} \sim P^0$ and recalling Proposition 3.2 one easily obtains

Proposition 3.18. *$\mathcal{M}^e(P^{\theta}) = \mathcal{M}^e(P_0)$ for all $\theta \in \Theta$. In addition, every $Q \in \mathcal{M}^e(P^{\theta})$ satisfies $dQ = \mathcal{E}\left(\lambda^{\theta} \cdot W^{\theta}\right) dP^{\theta}$ and $dQ = \mathcal{E}\left(\lambda^{0} \cdot W^{0}\right) dP_0$, with $\lambda^{\theta} = -\xi^{\theta} + \eta^{\theta}$ and $\lambda^{0} = -\xi^{0} + \eta^{0}$, where $\Pi^{\perp}(\lambda^{\theta}) = \eta^{\theta}, \quad \Pi^{\perp}(\lambda^{0}) = \eta^{0}$ and $\eta^{\theta} = \eta^{0} - \Pi^{\perp}(\theta)$.*

Thus, we simply write $\mathcal{M}^e = \mathcal{M}^e(S)$ for the set of equivalent martingale measures.

3.3.2 No-good-deal constraint and good-deal bounds under uncertainty

Let $\{C^{\theta} \mid \theta \in \Theta\}$ be a family of standard correspondences satisfying

$$-\xi^{\theta} \in C^{\theta} \quad \text{for all } \theta \in \Theta. \tag{3.40}$$

In the model P^{θ}, $\theta \in \Theta$, let the no-good-deal constraint be such that the Girsanov kernels of measures in $\mathcal{M}^e$ are selections of C^{θ}. The resulting set $\mathcal{Q}^{\mathrm{ngd}}(P^{\theta})$ of no-good-deal measures is equal to

$$\Big\{Q \sim P^{\theta} \;\Big|\; dQ/dP^{\theta} = \mathcal{E}(\lambda \cdot W^{\theta}),\ \lambda \text{ predictable, bounded, } \lambda \in (-\xi^{\theta} + \mathrm{Ker}\ \sigma) \cap C^{\theta}\Big\}.$$

By (3.40), then $\widehat{Q}^{\theta} \in \mathcal{Q}^{\mathrm{ngd}}(P^{\theta}) \neq \emptyset$ for all $\theta \in \Theta$. By Proposition 3.18 holds

$$\mathcal{Q}^{\mathrm{ngd}}(P^{\theta}) = \Big\{Q \sim P_0 \Big|\ dQ/dP_0 = \mathcal{E}(\lambda \cdot W^{0}),\quad \lambda \in -\xi^{0} + (\widetilde{C}^{\theta} \cap \mathrm{Ker}\ \sigma)\Big\} \tag{3.41}$$

where λ is predictable and bounded, and for all $\theta \in \Theta$ the correspondences $\widetilde{C}^{\theta}$ are given by

$$\widetilde{C}^{\theta} := C^{\theta} + \xi^{\theta} + \Pi^{\perp}(\theta) = C^{\theta} + \xi^{0} + \theta. \tag{3.42}$$

Following a worst-case approach, we take the (robust) upper good-deal valuation $\pi^{u}_{\cdot}(\cdot)$ under uncertainty as being the largest of all good-deal bounds $\pi^{u,\theta}_{\cdot}(\cdot)$ over all models $P^{\theta},\ \theta \in \Theta$. The respective set $\mathcal{Q}^{\mathrm{ngd}}$ of no-good-deal valuation measures corresponding to $\pi^{u}_{\cdot}(\cdot)$ can be described in terms of the sets $\mathcal{Q}^{\mathrm{ngd}}(P^{\theta}),\ \theta \in \Theta$. At first, one might guess that $\mathcal{Q}^{\mathrm{ngd}}$ should be the union of all $\mathcal{Q}^{\mathrm{ngd}}(P^{\theta})$. However, to have m-stability and convexity of $\mathcal{Q}^{\mathrm{ngd}}$ for good dynamic properties of the resulting good-deal bounds (as in Lemma 3.1), one has to define $\mathcal{Q}^{\mathrm{ngd}}$ as the smallest m-stable and convex set containing all $\mathcal{Q}^{\mathrm{ngd}}(P^{\theta}),\ \theta \in \Theta$.

Definition 3.19. *$\mathcal{Q}^{\mathrm{ngd}}$ is the smallest m-stable convex subset of $\mathcal{M}^e$ containing all $\mathcal{Q}^{\mathrm{ngd}}(P^{\theta})$, $\theta \in \Theta$. For sufficiently integrable claims X (e.g. in L^{∞}), the worst-case upper good-deal bound under uncertainty is $\pi^{u}_{t}(X) :=$ ess $\sup_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E^{Q}_{t}[X]$.*

We characterize the set $\mathcal{Q}^{\mathrm{ngd}}$ from Definition 3.19 using a suitable single correspondence $\widetilde{C}$ which is derived from all $C^{\theta},\ \theta \in \Theta$. To this end, we impose the

Assumption 3.20. *The correspondence with values* $\bigcup_{\theta\in\Theta} \widetilde{C}^{\theta}_t(\omega),\ (t,\omega) \in [0,T]\times\Omega$, *is compact-valued and predictable.*

The theory of measurable correspondences is well-developed for closed-valued correspondences (see [Roc76]). Assumption 3.20 ensures closed-valuedness and predictability of $\widetilde{C}$ for the proposition below. If all C^{θ} $(\theta\in\Theta)$ are equal to some given C^0, as in the following example, such an assumption will automatically hold in the setting required for Section 3.3.4, where $\widetilde{C}^{\theta}$ $(\theta\in\Theta)$ are ellipsoidal.

Example 3.21. *For a standard correspondence C^0 with $\xi^{\theta}\in C^0$, $\theta\in\Theta$, let $C^{\theta} := C^0$, $\theta\in\Theta$. Then $\widetilde{C}^{\theta} = C^0+\xi^0+\theta$ and $\bigcup_{\theta\in\Theta}\widetilde{C}^{\theta} = C^0+\xi^0+\Theta$ satisfies Assumption 3.20.*

Proposition 3.22. *Let Assumption 3.20 hold. Then $\mathcal{Q}^{\mathrm{ngd}}$ equals*

$$\Big\{Q\sim P_0 \;\Big|\; dQ/dP_0 = \mathcal{E}(\lambda\cdot W^0),\ \lambda = -\xi^0+\eta \textit{ predictable, bounded, } \eta\in\widetilde{C}\Big\}, \tag{3.43}$$

for the standard correspondence $\widetilde{C}_t(\omega) := \mathrm{Ker}\ \sigma_t(\omega)\cap \mathrm{Conv}\Big(\bigcup_{\theta\in\Theta}\widetilde{C}^{\theta}_t(\omega)\Big)$.

Proof. With Assumption 3.20, [Roc76, Theorem 1.M and Proposition 1.H] imply that $\widetilde{C}$ is standard. Note that $\widetilde{C}$ is non-empty-valued since $-\xi^0\in C^0$ and hence $0\in\widetilde{C}^0_t(\omega)\cap\mathrm{Ker}\ \sigma_t(\omega)\subset\widetilde{C}_t(\omega)$. Denote by $\mathcal{Q}$ the set in (3.43). By definition $\widetilde{C}_t(\omega)\subset\mathrm{Ker}\ \sigma_t(\omega)$, implying $\mathcal{Q}\subseteq\mathcal{M}^e$. We first prove that $\mathcal{Q}^{\mathrm{ngd}}\subseteq\mathcal{Q}$. Applying [Del06, Theorem 1] or following the steps of the proof for Lemma 3.1, Part b), one sees that $\mathcal{Q}$ is m-stable and convex. By (3.41) and since $\widetilde{C}^{\theta}_t(\omega)\cap\mathrm{Ker}\ \sigma_t(\omega)\subseteq\widetilde{C}_t(\omega)$ for all $\theta\in\Theta$, then $\mathcal{Q}$ contains the union of all $\mathcal{Q}^{\mathrm{ngd}}(P^{\theta})$, $\theta\in\Theta$. By definition $\mathcal{Q}^{\mathrm{ngd}}$ is the smallest m-stable convex subset of $\mathcal{M}^e$ with this property, hence $\mathcal{Q}^{\mathrm{ngd}}\subseteq\mathcal{Q}$.

Let us show $\mathcal{Q}\subseteq\mathcal{Q}^{\mathrm{ngd}}$. The L^1-closure of $\mathcal{Q}^{\mathrm{ngd}}$ is an m-stable closed and convex set of measures $Q\ll P_0$, and $\mathcal{Q}^{\mathrm{ngd}}$ comprises exactly those elements of its closure that are equivalent to P_0. Closeness and convexity of the closure of $\mathcal{Q}^{\mathrm{ngd}}$ are clear. We now show its m-stability. To this end, let Z^1_T, Z^2_T be in the closure of $\mathcal{Q}^{\mathrm{ngd}}$, $\tau\le T$ be a stopping time and $Z_T := Z^1_\tau Z^2_T/Z^2_\tau I_{\{Z^2_\tau>0\}} + Z^1_\tau I_{\{Z^2_\tau=0\}}$. There exist $\big(Z^{1,n}_T\big)_n, \big({}_nZ^{2,n}_T\big)_n\subseteq\mathcal{Q}^{\mathrm{ngd}}$ such that $Z^{1,n}_T\to Z^1_T$ and $Z^{2,n}_T\to Z^2_T$ in L^1. By m-stability of $\mathcal{Q}^{\mathrm{ngd}}$ holds $Z^n_T := Z^{1,n}_\tau Z^{2,n}_T/Z^{2,n}_\tau\in\mathcal{Q}^{\mathrm{ngd}}$ for each $n\in\mathbb{N}$. Now $E[Z^n_T]=1$ for all $n\in\mathbb{N}$, and $Z^n_T\to Z_T$ in probability as $n\to\infty$. In addition,

$$\begin{aligned}E[Z_T] &= E[Z^1_\tau Z^2_T/Z^2_\tau I_{\{Z^2_\tau>0\}}] + E[Z^1_\tau I_{\{Z^2_\tau=0\}}]\\ &= E\big[E_\tau[Z^2_T/Z^2_\tau]\ Z^1_\tau I_{\{Z^2_\tau>0\}}\big] + E[Z^1_\tau I_{\{Z^2_\tau=0\}}] = E[Z^1_\tau] = 1.\end{aligned}$$

By Scheffé's lemma one obtains $Z_T^n \to Z_T$ in L^1 as $n \to \infty$, and m-stability of the closure of $\mathcal{Q}^{\mathrm{ngd}}$ follows. As W^0 is a continuous P_0-martingale with the predictable representation property, it satisfies the hypotheses of [Del06, Theorem 2], implying by Definition 3.19 the existence of a closed-convex-valued predictable correspondence C^1 such that the no-good-deal measure set $\mathcal{Q}^{\mathrm{ngd}}$ is equal to

$$\Big\{Q \sim P_0 \Big| \, dQ/dP_0 = \mathcal{E}\left(\lambda \cdot W^0\right), \ \lambda = -\xi^0 + \eta \text{ predictable}, \, \eta \in C^1 \cap \mathrm{Ker}\ \sigma\Big\}.$$

To prove the claim, it suffices to show that all predictable selections of $\widetilde{C}$ are also predictable selections of $C^1 \cap \mathrm{Ker}\ \sigma$. To this end it suffices to show that for all $\theta \in \Theta$, any predictable selection of $\widetilde{C}^\theta \cap \mathrm{Ker}\ \sigma$ is a predictable selection of $C^1 \cap \mathrm{Ker}\ \sigma$. Assume the contrary that there exists $\theta \in \Theta$ and a predictable process η such that $\eta \in \widetilde{C}^\theta \cap \mathrm{Ker}\ \sigma$ and η is not selection of $C^1 \cap \mathrm{Ker}\ \sigma$. Then $\mathcal{E}\left((-\xi^0 + \eta) \cdot W^0\right)$ is in $\mathcal{Q}^{\mathrm{ngd}}(P^\theta)$ but not in $\mathcal{Q}^{\mathrm{ngd}}$, which contradicts $\mathcal{Q}^{\mathrm{ngd}}(P^\theta) \subseteq \mathcal{Q}^{\mathrm{ngd}}$.

$\square$

Using the characterization of $\mathcal{Q}^{\mathrm{ngd}}$ in Proposition 3.22 we can apply the results of Sections 3.1-3.2 in order to derive worst-case good-deal bounds and hedging strategies under uncertainty like in the absence of uncertainty, with the center P_0 of the set of reference measures $\mathcal{R}$ taking the role of P (in Sections 3.1-3.2) and the enlarged correspondence $\widetilde{C}$ taking the role of C there.

Example 3.23. *For $C^\theta, \theta \in \Theta$, as in Example 3.21 holds $\widetilde{C} = (C^0 + \xi^0 + \Theta) \cap \mathrm{Ker}\ \sigma$ and*

$$\mathcal{Q}^{\mathrm{ngd}} = \Big\{Q \sim P_0 \ \Big| \ dQ/dP_0 = \mathcal{E}\left(\lambda \cdot W^0\right), \ \lambda \in (-\xi^0 + \mathrm{Ker}\ \sigma) \cap (C^0 + \Theta)\Big\} \tag{3.44}$$

with λ denoting bounded predictable selections, by Proposition 3.22. Moreover the union $\bigcup_{\theta \in \Theta} \mathcal{Q}^{\mathrm{ngd}}(P^\theta)$ is convex, m-stable (cf. Lemma 3.26) and equals $\mathcal{Q}^{\mathrm{ngd}}$.

Remark 3.24. *a) Equation (3.43) shows, how the good-deal valuation and hedging problem under model uncertainty can technically be embedded into the mathematical framework of Sections 3.1-3.2 without uncertainty, by considering an enlarged no-good-deal constraint correspondence C as $\mathrm{Conv}(\cup_{\theta\in\Theta}(C^\theta + \theta))$ in (3.6) with P_0 taking the role of P. In Example 3.23, (3.44), it simply means to take C as $C^0 + \Theta$.*

b) Typical examples for good-deal constraints are radial, i.e. C^0 is a ball. This case is predominant in the literature and justified from a finance point of view by ensuring a constant bound on instantaneous Sharpe ratios (or growth rates). But typical examples for uncertainty (ambiguity) constraints Θ can well be non-radial (see [CE02, ES03]). For instance, Θ may arise from a confidence region for some unknown drift parameters in a multivariate (log-)normal model; such would in general be ellipsoidal but not radial, and the sum $C^0 + \Theta$ can even be *non-ellipsoidal. To offer a suitable framework for such and other examples, Section 3.1 treats abstract correspondences. A constructive method to solve for such a typical parametrization of* $C^0 + \Theta$ *is described in Remark 3.31.*

3.3.3 Robust approach to good-deal hedging under model uncertainty

As in Section 3.2 (cf. (3.15) and the definition of $\mathcal{Q}^{\mathrm{ngd}}(P^\theta)$), we define for $\theta \in \Theta$ the set $\mathcal{P}^{\mathrm{ngd}}(P^\theta) := \left\{ Q \sim P^\theta \mid dQ/dP^\theta = \mathcal{E}\left(\lambda \cdot W^\theta\right),\ \lambda \in C^\theta \text{ predictable, bounded} \right\}$ in order to introduce a robust notion of good-deal hedging. Let $\mathcal{P}^{\mathrm{ngd}}$ denote the smallest m-stable convex set of measures $Q \sim P_0$ containing all $\mathcal{P}^{\mathrm{ngd}}(P^\theta)$, $\theta \in \Theta$. Then

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q \in \mathcal{P}^{\mathrm{ngd}}} E_t^Q[X], \quad t \in [0,T],\ X \in L^2(P_0),$$

defines a time-consistent dynamic coherent risk measure by Lemma 3.1. Like in Section 3.2, the good-deal hedging problem under uncertainty is posed as a minimization problem (3.45) of a-priori risk measures ρ of hedging errors: for a contingent claim X, find a strategy $\phi^* \in \Phi$ such that for all $t \in [0,T]$ holds

$$\pi_t^u(X) = \rho_t\Big(X - \int_t^T \phi_s^{*\mathrm{tr}} d\widehat{W}_s^0\Big) = \operatorname*{ess\,inf}_{\phi \in \Phi}\ \rho_t\Big(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s^0\Big). \tag{3.45}$$

The good-deal hedging strategy under uncertainty is defined as this minimizer (if it exists) $\phi^* \in \Phi$. For $X \in L^2(P_0)$, one can prove (as in Proposition 3.12) that the tracking error $R^{\phi^*}(X)$ (defined as in (3.18)) of the strategy ϕ^* is a supermartingale under every measure in $\mathcal{P}^{\mathrm{ngd}}$:

Proposition 3.25. *For* $X \in L^2(P_0)$*, let* ϕ^* *be the strategy solving (3.45). Then the tracking error* $R^{\phi^*}(X)$ *of this strategy is a* Q*-supermartingale for all* $Q \in \mathcal{P}^{ngd}$*.*

A strategy solving the good-deal hedging problem under uncertainty and whose tracking error satisfies the supermartingale property under all measures in $\mathcal{P}^{\mathrm{ngd}}$ (as in Proposition 3.25) will be qualified as *robust* with respect to uncertainty. Note that this is a different notion of robustness compared to the one in Remark 2.12, because the supermartingale property has to hold for measures in $\mathcal{P}^{\mathrm{ngd}}(P^\theta)$ uniformly for all models $P^\theta \in \mathcal{R}$ (since $\bigcup_{\theta\in\Theta} \mathcal{P}^{\mathrm{ngd}}(P^\theta)$ is a subset of $\mathcal{P}^{\mathrm{ngd}}$). More concrete results under uncertainty will be derived next under additional conditions.

3.3.4 Hedging under model uncertainty for ellipsoidal good-deal constraints

In this section we consider ellipsoidal good-deal constraints. To this end, let

$$C_t^0(\omega) = \left\{ x \in \mathbb{R}^n \;\middle|\; x^{\mathrm{tr}} A_t(\omega) x \le h_t^2(\omega) \right\}, \quad (t,\omega) \in [0,T] \times \Omega, \tag{3.46}$$

where A is a uniformly elliptic and predictable matrix-valued process, and h some positive bounded and predictable process. We assume that A satisfies the separability condition (3.22) with respect to σ. Let Θ be an arbitrary standard correspondence satisfying the uniform boundedness Assumption 3.3 and $0 \in \Theta$. As in Example 3.21, we let $C^\theta := C^0$, for all $\theta \in \Theta$, yielding by (3.42) that

$$\widetilde{C}_t^\theta \cap \mathrm{Ker}\ \sigma_t = \Big\{ x \in \mathbb{R}^n \Big| \ x^{\mathrm{tr}} A_t x \le h_t^2 - \xi_t^{\theta\,\mathrm{tr}} A_t \xi_t^\theta \Big\} \bigcap \mathrm{Ker}\ \sigma_t + \Pi_t^\perp(\theta_t).$$

Clearly, C^θ is standard and satisfies Assumption 3.3 for $\theta \in \Theta$. Similarly to (3.27), to derive explicit BSDE formulations for solving the hedging problem we will assume

$$|\xi^\theta| < h\sqrt{\alpha'} \quad \text{for all } \theta \in \Theta, \tag{3.47}$$

where the process α' is the constant of ellipticity of A^{-1} as in Lemma 3.16. Recall that, thanks to Lemma 3.16, the inequality (3.47) implies in particular that $-\xi^\theta \in C^0,\ \theta \in \Theta$; hence (3.40) holds and the correspondences $\widetilde{C}^\theta \cap \mathrm{Ker}\ \sigma$ are standard, $\theta \in \Theta$. Note that condition (3.47) ensures applicability of Lemma 3.35 in our current setup for any model P^θ. Since C^θ is equal to C^0 and satisfies Assumption 3.3, one has

$$\mathcal{P}^{\mathrm{ngd}}(P^\theta) = \Big\{ Q \sim P_0 \ \Big| \ dQ/dP_0 = \mathcal{E}(\lambda \cdot W^0),\ \lambda \text{ predictable},\ \lambda \in C^0 + \theta \Big\}. \tag{3.48}$$

The following lemma is proven in the Appendix.

Lemma 3.26. *1. The set $\bigcup_{\theta\in\Theta} \mathcal{P}^{\mathrm{ngd}}(P^\theta)$ is m-stable, convex and equal to $\mathcal{P}^{\mathrm{ngd}}$.*

2. The set $\bigcup_{\theta\in\Theta} \mathcal{Q}^{\mathrm{ngd}}(P^\theta)$ is m-stable, convex and equal to $\mathcal{Q}^{\mathrm{ngd}}$.

Thanks to Lemma 3.26, the dynamic risk measure ρ satisfies for $X \in L^2(P_0)$

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{P}^{\mathrm{ngd}}} E_t^Q[X] = \operatorname*{ess\,sup}_{\theta\in\Theta} \rho_t^\theta(X), \quad t \in [0,T],$$

with $\rho_t^\theta(X) := \operatorname{ess\,sup}_{Q\in\mathcal{P}^{\mathrm{ngd}}(P^\theta)} E_t^Q[X]$. The worst-case upper good-deal bound $\pi_t^u(X)$ for $X \in L^2(P_0)$ rewrites from Definition 3.19 as

$$\pi_t^u(X) := \operatorname*{ess\,sup}_{\theta\in\Theta} \ \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}(P^\theta)} E_t^Q[X] = \operatorname*{ess\,sup}_{\theta\in\Theta} \pi_t^{u,\theta}(X),\ t \in [0,T], \tag{3.49}$$

where $\pi_t^{u,\theta}(X) = \operatorname{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}(P^\theta)} E_t^Q[X]$. The corresponding lower bound $\pi_\cdot^l(X)$ is obtained via $\pi_\cdot^l(X) = -\pi_\cdot^u(-X)$. For a worst-case approach to uncertainty we will investigate valuation of claims according to $\pi_\cdot^u(\cdot)$ and hedging with the optimal trading strategy solution to (3.45). We employ results from Section 3.2.1 (under $P = P^\theta$) to characterize $\pi_\cdot^{u,\theta}(X)$ as well as the associated hedging strategies $\bar{\phi}^\theta$ in Φ. For $\theta \in \Theta$ and $\phi \in \Phi$ let us consider the classical BSDEs

$$-dY_t = f^{\phi,\theta}(t, Z_t)dt - Z_t^{\mathrm{tr}} dW_t^0, \quad t \le T, \quad Y_T = X \quad \text{and} \tag{3.50}$$

$$-dY_t = f^{\theta}(t, Z_t)dt - Z_t^{\mathrm{tr}} dW_t^0, \quad t \le T, \quad Y_T = X, \tag{3.51}$$

with generators

$$f^{\phi,\theta}(t,z) = \theta_t^{\mathrm{tr}}(z-\phi_t) - \xi_t^{0\,\mathrm{tr}}\phi_t + h_t\big((z-\phi_t)^{\mathrm{tr}}A_t^{-1}(z-\phi_t)\big)^{1/2} \tag{3.52}$$

$$\begin{aligned} f^{\theta}(t,z) = \quad & \Pi_t^{\perp}(\theta_t)^{\mathrm{tr}}\Pi_t^{\perp}(z) - \xi_t^{0\,\mathrm{tr}}\Pi_t(z) \\ & + \big(h_t^2 - \xi_t^{\theta\,\mathrm{tr}}A_t\xi_t^{\theta}\big)^{1/2}\big(\Pi_t^{\perp}(z)^{\mathrm{tr}}A_t^{-1}\Pi_t^{\perp}(z)\big)^{1/2}. \end{aligned} \tag{3.53}$$

It is straightforward to derive the BSDE descriptions for $\pi^{u,\theta}_\cdot(X)$ and $\rho^{\theta}_\cdot(X)$ stated in the subsequent proposition. The proof is analogous to that for Theorem 3.17, using (3.47) instead of (3.27), replacing P by P^θ and changing measure from P^θ to P_0.

Proposition 3.27. *Assume (3.22) and (3.47) hold. For $X \in L^2(P_0)$, $\theta \in \Theta$ and $\phi \in \Phi$, let $(Y^{\phi,\theta}, Z^{\phi,\theta})$ and (Y^θ, Z^θ) be the standard solutions to the BSDEs (3.50) and (3.51) respectively. Then $\pi_t^{u,\theta}(X) = Y_t^\theta = E_t^{\bar{Q}^\theta}[X],\ t\in[0,T]$, holds with $\bar{Q}^\theta \in \mathcal{Q}^{\mathrm{ngd}}(P^\theta)$ given by $d\bar{Q}^\theta/dP_0 = \mathcal{E}\left((-\xi^0+\bar{\eta}^\theta)\cdot W^0\right)$ for*

$$\bar{\eta}_t^\theta = \big(h_t^2 - \xi_t^{\theta\,tr}A_t\xi_t^\theta\big)^{1/2}\big(\Pi_t^\perp(Z_t^\theta)^{tr}A_t^{-1}\Pi_t^\perp(Z_t^\theta)\big)^{-1/2}A_t^{-1}\Pi_t^\perp(Z_t^\theta) + \Pi_t^\perp(\theta_t).$$

Moreover $Y_t^{\phi,\theta} = \rho_t^\theta\left(X - \int_t^T \phi_s^{tr}d\widehat{W}_s^0\right)$ holds, and the strategy $\bar{\phi}^\theta$ (in Φ)

$$\bar{\phi}_t^\theta := \Pi_t(Z_t^\theta) + \big(\Pi_t^\perp(Z_t^\theta)^{tr}A_t^{-1}\Pi_t^\perp(Z_t^\theta)\big)^{1/2}\big(h_t^2 - \xi_t^{\theta\,tr}A_t\xi_t^\theta\big)^{-1/2}A_t\xi_t^\theta$$

satisfies $\pi_t^{u,\theta}(X) = \rho_t^\theta\Big(X - \displaystyle\int_t^T (\bar{\phi}_s^\theta)^{tr}d\widehat{W}_s^0\Big) = \operatorname*{ess\,inf}_{\phi\in\Phi}\rho_t^\theta\Big(X - \displaystyle\int_t^T \phi_s^{tr}d\widehat{W}_s^0\Big)$.

By Proposition 3.27, we can write $\pi^u_\cdot(X)$ from (3.49) as

$$\pi_t^u(X) = \operatorname*{ess\,sup}_{\theta\in\Theta}\ \operatorname*{ess\,inf}_{\phi\in\Phi}\rho_t^\theta\Big(X - \int_t^T \phi_s^{\mathrm{tr}}d\widehat{W}_s^0\Big), \qquad t\in[0,T]. \tag{3.54}$$

This permits to describe $\pi^u_\cdot(X)$ and the associated hedging strategy $\bar{\phi}$ in the next theorem by the solution to the classical BSDE

$$-dY_t = f(t,Z_t)dt - Z_t^{\mathrm{tr}}dW_t^0,\ t\le T \quad \text{and} \quad Y_T = X, \tag{3.55}$$

with generator $f(t,Z_t) := \operatorname{ess\,sup}_{\theta\in\Theta} f^\theta(t,Z_t)$, for f^θ given in (3.53). The theorem moreover identifies by $\bar{\theta}$ the worst-case model $P^{\bar{\theta}} \in \mathcal{R}$.

Theorem 3.28. *Assume (3.22) and (3.47) hold. For $X \in L^2(P_0)$, let (Y,Z) be the standard solution to the BSDE (3.55). Then there exists a unique predictable selection $\bar{\theta} := \bar{\theta}(X)$ of Θ satisfying $\bar{\theta}_t = \operatorname{argmax}_{\theta\in\Theta} f^\theta(t,Z_t)$ such that for all $t\in[0,T]$*

$$\pi_t^u(X) = \rho_t^{\bar{\theta}}\Big(X - \int_t^T \bar{\phi}_s^{tr}d\widehat{W}_s^0\Big) = \pi_t^{u,\bar{\theta}}(X) = Y_t \tag{3.56}$$

holds with $\bar\phi = (\bar\phi_t)_{t\in[0,T]} := \bar\phi^{\bar\theta}(X) \in \Phi$ *given by*

$$\bar\phi_t = \Pi_t(Z_t) + \left(\Pi_t^\perp(Z_t)^{tr} A_t^{-1} \Pi_t^\perp(Z_t)\right)^{1/2} \left(h_t^2 - \xi_t^{\bar\theta^{tr}} A_t \xi_t^{\bar\theta}\right)^{-1/2} A_t \xi_t^{\bar\theta}]. \qquad (3.57)$$

The tracking error $R^{\bar\phi}(X) := \pi_\cdot^u(X) - \pi_0^u(X) - \bar\phi \cdot \widehat{W}^0$ *of the strategy* $\bar\phi$ *is a supermartingale under any* Q *in* $\mathcal{P}^{ngd}(P^{\bar\theta})$*, and is a martingale under* $\bar Q$ *in* $\mathcal{P}^{ngd}(P^{\bar\theta})$ *given by* $d\bar Q/dP_0 = \mathcal{E}(\bar\lambda \cdot W^0)$ *with*

$$\bar\lambda_t := h_t\big((Z_t - \bar\phi_t)A_t^{-1}(Z_t - \bar\phi_t)\big)^{-1/2} A_t^{-1}\left(Z_t - \bar\phi_t\right) + \bar\theta_t, \quad t \in [0,T].$$

Proof. Pointwise existence and uniqueness of $\bar\theta \in \Theta$ follow by the continuity and strict concavity of f^θ as a function of $\theta \in \mathbb{R}^n$, and the uniform boundedness of Θ. Predictability of $\bar\theta$ follows by [Roc76]. The claims (3.56) and (3.57) are corollaries of Proposition 3.27. The remaining claims are similar to those of Theorem 3.17, hence their proof goes likewise, making again use of Lemma 3.35 (instead of [Bec09, Lemma 6.1]) and (3.22) and (3.47). □

The process $\bar\phi := \bar\phi^{\bar\theta}$ in Theorem 3.28 is the good-deal hedging strategy of X for the worst-case model $P^{\bar\theta} \in \mathcal{R}$ which yields that highest good-deal valuation with $\pi_\cdot^u(X) = \pi_\cdot^{u,\bar\theta}(X)$. The tracking error of $\bar\phi$ is therefore a supermartingale under any measure in $\mathcal{P}^{\mathrm{ngd}}(P^{\bar\theta})$ (cf. Proposition 3.12), i.e. $\bar\phi$ is "at least mean-self-financing" under any measure in $\mathcal{P}^{\mathrm{ngd}}(P^{\bar\theta})$. However, it is not clear at this stage whether the supermartingale property of the tracking error of $\bar\phi$ holds simultaneously under all measures in $\mathcal{P}^{\mathrm{ngd}}(P^\theta)$ for all models $\mathcal{R} = \{P^\theta : \theta \in \Theta\}$. We will show that this is the case, and that $\bar\phi$ and its associated valuation bound $\pi_\cdot^u(X)$ are indeed robust with respect to uncertainty. The idea is first to find an alternative bound $\pi_\cdot^{u,*}(\cdot)$ and an associated strategy ϕ^* that satisfy the supermartingale property of the tracking error simultaneously under all measures in $\bigcup_{\theta\in\Theta} \mathcal{P}^{\mathrm{ngd}}(P^\theta)$ and are therefore robust. After this, we show that $\pi_\cdot^{u,*}(X)$ coincides with the worst-case bound $\pi_\cdot^u(X)$, and that the same holds for the hedging strategies $\bar\phi(X)$ and $\phi^*(X)$ for any contingent claim X. In general the good-deal bound $\pi_\cdot^u(X)$ is dominated by $\pi_\cdot^{u,*}(X)$, but thanks to a saddle point result (Theorem 3.30) one can actually prove that the two bounds are identical. Exchanging the order between ess sup and ess inf in the expression (3.54) for $\pi_\cdot^u(X)$, we define for $X \in L^2(P_0)$ and $t \le T$

$$\pi_t^{u,*}(X) := \operatorname*{ess\,inf}_{\phi\in\Phi} \operatorname*{ess\,sup}_{\theta\in\Theta} \rho_t^\theta\Big(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s^0\Big). \qquad (3.58)$$

From this it is clear that in general $\pi_t^{u,*}(X) \ge \pi_t^u(X)$, for all $X \in L^2(P_0)$. We will show that in fact the minimax identity holds in the sense that the expressions in (3.54) and (3.58) coincide, and that a saddle point exists, giving equality of $\pi_\cdot^u(X)$ and $\pi_\cdot^{u,*}(X)$. To this end, we describe $\pi_\cdot^{u,*}(X)$ and ϕ^* in terms of the standard solution (Y,Z) for the BSDE

$$-dY_t = f^*(t, Z_t)dt - Z_t^{\mathrm{tr}} dW_t^0, \quad t \le T \quad \text{and} \quad Y_T = X, \qquad (3.59)$$

where $f^*(t,Z_t) := \text{ess inf}_{\phi\in\Phi} f^\phi(t,Z_t)$, with $f^\phi(t,Z_t) := \text{ess sup}_{\theta\in\Theta} f^{\phi,\theta}(t,Z_t)$ for $f^{\phi,\theta}$ from (3.52). Indeed

$$f^\phi(t,Z_t) := -\xi_t^{0\,\mathrm{tr}}\phi_t + \operatorname*{ess\,sup}_{\theta_t\in\Theta_t} \theta_t^{\mathrm{tr}}(Z_t-\phi_t) + h_t\big((Z_t-\phi_t)^{\mathrm{tr}}A_t^{-1}(Z_t-\phi_t)\big)^{1/2} \qquad (3.60)$$

holds and we can identify the robust good-deal hedging strategy ϕ^* by

Proposition 3.29. *Assume (3.22) and (3.47) hold. For $X\in L^2(P_0)$, let (Y,Z) be the standard solution to the BSDE (3.59). Then there exists a unique $\phi^*\in\Phi$ satisfying $\phi_t^* = \text{argmin}_{\phi\in\Phi} f^\phi(t,Z_t)$ for $t\in[0,T]$ such that*

$$\pi_t^{u,*}(X) = \operatorname*{ess\,inf}_{\phi\in\Phi}\rho_t\Big(X-\int_t^T \phi_s^{tr}d\widehat{W}_s^0\Big) = \rho_t\Big(X-\int_t^T \phi_s^{*\,tr}d\widehat{W}_s^0\Big) = Y_t,\ t\in[0,T]. \qquad (3.61)$$

Moreover $R^{\phi^*}(X) := \pi_\cdot^{u,*}(X)-\pi_0^{u,*}(X)-\phi^*\cdot\widehat{W}^0$ is a Q-supermartingale for all $Q\in\mathcal{P}^{ngd}$, and a Q^*-martingale for $Q^*\in\mathcal{P}^{ngd}$ with $dQ^*/dP_0=\mathcal{E}(\lambda^*\cdot W^0)$, where

$$\lambda^* = h\big((Z-\phi^*)^{tr}A^{-1}(Z-\phi^*)\big)^{-1/2}A^{-1}(Z-\phi^*)+\theta^*,$$

with $\theta_t^* := \theta_t^*(\phi^*) = \text{argmax}_{\theta\in\Theta}\theta_t^{tr}(Z_t-\phi_t^*)$ such that $f^*(t,Z_t)=f^{\phi^*,\theta^*}(t,Z_t)$.

Proof. It is clear that for any $\phi\in\Phi$, there exists $\theta^*(\phi)\in\Theta$ such that $\theta_t^*(\phi)^{\mathrm{tr}}(Z_t-\phi_t) = \text{ess sup}_{\theta_t\in\Theta_t}\theta_t^{\mathrm{tr}}(Z_t-\phi_t)$ and $f^\phi(t,Z_t) = f^{\phi,\theta^*(\phi)}(t,Z_t)$. Consider the convex continuous function $\mathbb{R}^n\ni\phi\mapsto F(\phi) := -\xi^{0\,\mathrm{tr}}\phi + \text{ess sup}_{\theta\in\Theta}\theta^{\mathrm{tr}}(z-\phi)+h\big((z-\phi)^{\mathrm{tr}}A^{-1}(z-\phi)\big)^{1/2}$, for constant h,ϕ,z,ξ^0,σ and A satisfying the notations of Lemma 3.35 and for a compact set $\Theta\subset\mathbb{R}^n$ containing the origin. The function F is also coercive on $\mathrm{Im}\ \sigma^{\mathrm{tr}}$, i.e. $F(\phi)\to+\infty$ as $|\phi|\to+\infty$ for $\Pi^\perp(\phi)=0$ because $|\xi^0|<h\sqrt{\alpha'}$ and $\text{ess sup}_{\theta\in\Theta}\theta^{\mathrm{tr}}(z-\phi)\geq 0$. Hence existence of $\phi^*\in\Phi$ follows from [ET99, Chapter II, Proposition 1.2]. Uniqueness of ϕ^* follows from the fact that F is strictly convex over $\big\{\Pi^\perp(\phi)=0\big\}$ if $\Pi^\perp(z)\neq 0$ and strictly convex at $\phi=z$ if $\Pi^\perp(z)=0$ because (3.47) holds. Finally, predictability of ϕ^* follows from [Roc76, Theorem 2.K] via Part 1 of Proposition 3.2.

From Proposition 3.27, for $\phi\in\Phi$ and $\theta\in\Theta$, $Y^{\phi,\theta}=\rho_\cdot^\theta\big(X-\int_\cdot^T\phi_s^{\mathrm{tr}}d\widehat{W}_s^0\big)$ is the Y-component of the solution to the classical BSDE (3.50). As a consequence for every $\phi\in\Phi$ it holds $\text{ess sup}_{\theta\in\Theta} f^{\phi,\theta}(t,Z_t) = f^{\phi,\theta^*(\phi)}(t,Z_t) = f^\phi(t,Z_t)$, $t\in[0,T]$. The generators f^ϕ are standard, so that by the comparison theorem for classical BSDEs, (Y^ϕ,Z^ϕ) with $Y_t^\phi := \text{ess sup}_{\theta\in\Theta} Y_t^{\phi,\theta}$ is the standard solution to the BSDEs (under P_0) with parameters (f^ϕ,X), for $\phi\in\Phi$. The generator f^* is also standard because $f^*(t,Z_t)=f^{\phi^*,\theta^*(\phi^*)}(t,Z_t)=\text{ess inf}_{\phi\in\Phi}f^\phi(t,Z_t)$. Now the comparison theorem yields (3.61) from (3.58).

The supermartingale property of $R^{\phi^*}(X)$ can be proved from (3.61) using arguments in the proof of Proposition 3.12. A BSDE proof for the supermartingale property can also be given along the same line as the following for the martingale property. By (3.48) it holds $Q^* \in \mathcal{P}^{\mathrm{ngd}}(P^{\theta^*}) \subset \mathcal{P}^{\mathrm{ngd}}$ since $\lambda^* \in C^0 + \theta^*$ and $\theta^* \in \Theta$. Because $\pi^{u,*}(X)$ is the value process of the BSDE (3.59), then after changing measures from P_0 to Q^* one obtains

$$-dR_t^{\phi^*}(X) = (f^*(t,Z_t) + \xi_t^{0\,\mathrm{tr}}\phi_t^* - \lambda_t^{*\mathrm{tr}}(Z_t - \phi_t^*))dt - (Z_t - \phi_t^*)^{\mathrm{tr}} dW_t^{Q^*}. \qquad (3.62)$$

Furthermore the finite variation part of (3.62) vanishes since $f^*(t,Z_t) = f^{\phi^*,\theta^*}(t,Z_t)$. Because $R^{\phi^*}(X) \in \mathcal{S}^2(P_0)$ and $dQ^*/dP_0 \in L^p(P_0)$ for all $p < \infty$ (since λ^* is bounded), then Hölder's inequality implies that $R^{\phi^*}(X) \in \mathcal{S}^{2-\epsilon}(Q^*)$ for $\epsilon \in (0,1)$. Thus $R^{\phi^*}(X)$ is a Q^*-martingale.

□

Proposition 3.29 shows that the tracking error of the hedging strategy ϕ^* with respect to valuation according to $\pi_\cdot^{u,*}(X)$ has the supermartingale property simultaneously under all measures in $\mathcal{P}^{\mathrm{ngd}} = \bigcup_{\theta\in\Theta}\mathcal{P}^{\mathrm{ngd}}(P^\theta)$. The next theorem shows that a minimax identity holds: the sup-inf representation of $\pi_\cdot^u(\cdot)$ in (3.54) is equal to the inf-sup representation of $\pi_\cdot^{u,*}(\cdot)$ in (3.58); see also (3.63). Moreover, the good-deal hedging strategy $\bar{\phi}$ with respect to the worst-case model (given by $\bar{\theta}$) that gives the highest good-deal valuation bound $\pi_\cdot^u(\cdot)$, is identical with the robust good-deal hedging strategy ϕ^* from Proposition 3.29.

Theorem 3.30. *Assume (3.22) and (3.47) hold. For $X \in L^2(P_0)$, let (Y,Z) be standard solution of the BSDE (3.59). Then*

$$f^{\phi^*,\theta^*}(t,Z_t) = \operatorname*{ess\,inf}_{\phi\in\Phi}\ \operatorname*{ess\,sup}_{\theta\in\Theta} f^{\phi,\theta}(t,Z_t) = \operatorname*{ess\,sup}_{\theta\in\Theta}\ \operatorname*{ess\,inf}_{\phi\in\Phi} f^{\phi,\theta}(t,Z_t) = f^{\bar{\phi},\bar{\theta}}(t,Z_t) \qquad (3.63)$$

holds with $(\bar{\phi},\bar{\theta})$, (ϕ^*,θ^*) from Theorem 3.28 and Proposition 3.29. Moreover (Y,Z) coincides with the standard solution to the BSDE (3.55) and

$$\pi_t^u(X) = \pi_t^{u,*}(X) = Y_t \quad \textit{and} \quad \phi_t^*(X) = \bar{\phi}_t(X), \quad t \in [0,T]. \qquad (3.64)$$

Proof. Let $X \in L^2(P_0)$. By an application of Lemma 3.36, the generator $f^{\phi,\theta}$ of the BSDE (3.50) for $\theta \in \Theta$ and $\phi \in \Phi$ satisfy the minimax relation (3.63). By Theorem 3.28 and Proposition 3.29 it holds $f(t,Z_t) = f^{\bar{\phi},\bar{\theta}}(t,Z_t)$ and $f^*(t,Z_t) = f^{\phi^*,\theta^*}(t,Z_t),\ t \in [0,T]$, for f, f^* respectively generators of the BSDEs (3.55), (3.59). Also, $\pi_t^u(X) = \pi_t^{u,*}(X) = Y_t$, $t \in [0,T]$, since by uniqueness of BSDE solutions (Y,Z) also solves the BSDE (3.55). Hence $(\bar{\phi},\bar{\theta})$ and (ϕ^*,θ^*) are both saddle points of the function $(\phi_t,\theta_t) \mapsto f^{\phi,\theta}(t,Z_t)$. Now for any $\theta \in \Theta$ and $z \in \mathbb{R}^n$, the function $\phi \mapsto F(\phi,\theta) := \theta^{\mathrm{tr}}(z-\phi) - \xi^{0\,\mathrm{tr}}\phi + h\big((z-\phi)^{\mathrm{tr}}A^{-1}(z-\phi)\big)^{1/2}$ is strictly convex over $\{\Pi^\perp(\phi) = 0\}$ if $\Pi^\perp(z) \neq 0$, and strictly convex at $\phi = z$ if $\Pi^\perp(z) = 0$, since $\left|\xi^\theta\right| < h\sqrt{\alpha'}$. [ET99, Chapter VI, Proposition 1.5] implies that the ϕ-components of the saddle points are identical, yielding $\bar{\phi} = \phi^*$.

□

3.3.5 The impact of model uncertainty on robust good-deal hedging

In the framework of Section 3.3.4, results have so far been stated for an arbitrary standard correspondence Θ without further structural assumptions, and ellipsoidal correspondences were only assumed for the no-good-deal constraints C^θ, $\theta \in \Theta$. Recall (cf. Theorem 3.17 and subsequent remarks) that in the absence of uncertainty the good-deal hedging strategy contains a speculative component in the direction of the market price of risk. This already indicates that under uncertainty one should expect to see relevant differences by a robust approach to hedging. To investigate the effect of uncertainty about the market price of risk θ on robust good-deal hedging, we assume in addition (noting that $\theta \in \operatorname{Im}\sigma^{\mathsf{tr}}$ is natural) that for all $(t,\omega) \in [0,T]\times\Omega$, the set $\Theta_t(\omega)$ is a subset of $\operatorname{Im}\,\sigma_t^{\mathsf{tr}}(\omega)$ in the sense that

$$\Theta_t(\omega) = \Theta_t^0(\omega) \cap \operatorname{Im}\,\sigma_t^{\mathsf{tr}}(\omega) \tag{3.65}$$

holds for some standard correspondence Θ^0 with $0 \in \Theta^0$ satisfying the uniform boundedness Assumption 3.3. With (3.65), one clearly has $\Pi^\perp(\theta) = 0$ for all $\theta \in \Theta$. This leads to the following simplified expressions of the BSDE generators $f^{\phi,\theta}$, f^θ:

$$\begin{aligned} f^{\phi,\theta}(t,z) &= \theta_t^{\mathrm{tr}}(\Pi_t(z)-\phi_t) - \xi_t^{0\,\mathrm{tr}}\phi_t + h_t\big((z-\phi_t)^{\mathrm{tr}}A_t^{-1}(z-\phi_t)\big)^{1/2}, \text{ and} \\ f^{\theta}(t,z) &= -\xi_t^{0\,\mathrm{tr}}\Pi_t(z) + \big(h_t^2 - \xi_t^{\theta\,\mathrm{tr}}A_t\xi_t^\theta\big)^{1/2}\big(\Pi_t^\perp(z)^{\mathrm{tr}}A_t^{-1}\Pi_t^\perp(z)\big)^{1/2}. \end{aligned}$$

As a consequence, the process $\bar\theta = \bar\theta(X)$ does actually not depend on the contingent claim $X \in L^2(P_0)$ under consideration, and solves the minimization problem

$$\xi_t^{\bar\theta\ \mathrm{tr}}A_t\xi_t^{\bar\theta} = \min_{\theta_t\in\Theta_t} \xi_t^{\theta\,\mathrm{tr}}A_t\xi_t^\theta, \quad t\in[0,T]. \tag{3.66}$$

In addition in this case, one has $\mathcal{Q}^{\mathsf{ngd}}(P^{\bar\theta}) = \bigcup_{\theta\in\Theta}\mathcal{Q}^{\mathsf{ngd}}(P^\theta) = \mathcal{Q}^{\mathsf{ngd}}$. To obtain even more explicit results one may assume e.g. ellipsoidal uncertainty

$$\Theta_t^0(\omega) := \Big\{x\in\mathbb{R}^n \mid x^{\mathrm{tr}}B_t(\omega)x \le \delta_t^2(\omega)\Big\} \quad \text{for all } (t,\omega)\in[0,T]\times\Omega, \tag{3.67}$$

with δ being a positive bounded and predictable process, and B being a uniformly elliptic and predictable matrix-valued process, satisfying the separability condition (3.22) with respect to σ. Clearly $f^\phi(t,Z_t)$ from (3.60) in this case is equal to

$$-\xi_t^{0\,\mathrm{tr}}\phi_t + \delta_t\big((\Pi_t(Z_t)-\phi_t)^{\mathrm{tr}}B_t^{-1}(\Pi_t(Z_t)-\phi_t)\big)^{1/2} + h_t\big((Z_t-\phi_t)^{\mathrm{tr}}A_t^{-1}(Z_t-\phi_t)\big)^{1/2}.$$

In terms of ϕ^* and the solution (Y,Z) to the BSDE (3.59), the process $\theta^* = \theta^*(\phi^*)$ of Proposition 3.29 is given by

$$\theta_t^*(X) = \delta_t\big((\Pi_t(Z_t)-\phi_t^*)^{\mathrm{tr}}B_t^{-1}(\Pi_t(Z_t)-\phi_t^*)\big)^{-1/2}B_t^{-1}(\Pi_t(Z_t)-\phi_t^*). \tag{3.68}$$

Remark 3.31. *Let us recall Remark 3.24 b). In the present context of Section 3.3.5 with constraints of ellipsoidal type for good-deals (3.46) and for model uncertainty (3.67), results as explicit as in Section 3.2.1 can be obtained in particular cases, as elaborated subsequently, but not in general. Indeed, using $\xi^\theta = \xi^0 + \theta$ for $\theta \in \Theta$, to find the minimizer $\bar\theta$ (the worst-case) in (3.66) requires to compute the projection of $-\xi^0_t$ onto the ellipsoid Θ_t with respect to the norm induced by the matrix A_t. In the radial case $A \equiv Id_{\mathbb{R}^n}$ the projection is Euclidian. While there is no closed formula for the projection in general, the solution is described by a parametric formula in terms of a Lagrangian multiplier that solves a 1-dimensional equation, and it can be computed by efficient algorithms (see [Kis94]) even if this operation may be required frequently (as in Monte Carlo simulation, cf. Section 3.2.2).*

It is instructive to look at the special case where in addition the matrices A and B are related through $B = A/r$ for some scalar $r > 0$; in other words, B basically equals A up to a change of δ to $\sqrt{r}\delta$. In this case (3.66) is solved by

$$\bar\theta_t = -\xi^0_t I_{\{\xi^{0\,\mathrm{tr}}_t A_t \xi^0_t \le r\delta^2_t\}} - \frac{\sqrt{r}\delta_t}{(\xi^{0\,\mathrm{tr}}_t A_t \xi^0_t)^{1/2}} \xi^0_t I_{\{\xi^{0\,\mathrm{tr}}_t A_t \xi^0_t > r\delta^2_t\}}, \quad t \in [0,T], \tag{3.69}$$

and replacing $\phi^* = \bar\phi$ in the formula of θ^* in (3.68) by its expression from (3.57) in terms of $\bar\theta$ one obtains $\theta^* = \bar\theta$. Note that (3.69) implies that $\bar\theta^{\mathrm{tr}} A \bar\theta$ is equal to $\xi^{0\,\mathrm{tr}} A \xi^0$ on $\{\xi^{0\,\mathrm{tr}} A \xi^0 \le r\delta^2\}$ and equal to $r\delta^2$ on $\{\xi^{0\,\mathrm{tr}} A \xi^0 > r\delta^2\}$. In other words, the worst-case Girsanov kernel $-\bar\theta$ is equal to the market price of risk ξ^0 of the center P_0 of the confidence set $\mathcal{R}$ of reference measures, being truncated such that $\bar\theta^{\mathrm{tr}} A \bar\theta = r\delta^2$ holds for large values of ξ^0 outside of the ellipsoidal set $\{x \in \mathbb{R}^n : x^{\mathrm{tr}} A x \le r\delta^2\}$.

To obtain an intuition about the impact that model uncertainty may have on robust good-deal hedging, let us look at the behavior of the worst-case Girsanov kernel $\bar\theta = \theta^*$ obtained in (3.69) and the hedging strategy $\bar\phi = \phi^*$ in (3.57) for varying scaling constant r: As r becomes large, the worst-case Girsanov kernel $-\bar\theta$ becomes close to the market price of risk ξ^0 and $\phi^* = \bar\phi$ close to $\Pi(Z)$. This shows that as uncertainty becomes overwhelming, the robust good-deal hedging strategy ceases to comprise a speculative component in the direction of the market price of risk. In such a situation one can show that the hedging strategy is the risk-minimizing strategy under the worst-case no-good-deal measure in the worst-case model $P^{\bar\theta}$. More precisely, for an arbitrary shape of the correspondence Θ^0, if uncertainty is big enough for the confidence set $\mathcal{R}$ of reference measures to contain some risk neutral pricing measure from $\mathcal{M}^e$, then robust good-deal hedging for any claim X does not comprise a speculative component and the holdings ϕ^* of the hedging strategy in risky assets coincide with those of the globally risk-minimizing strategy by [FS86] (cf. also [Sch01, Section 2]) under worst-case no-good-deal measure $\bar Q = \bar Q(X, P^{\bar\theta}) \in \mathcal{Q}^{\mathrm{ngd}}(P^{\bar\theta})$, i.e. satisfying $\pi^u_t(X) = \pi^{u,\bar\theta}_t(X) = E^{\bar Q}_t[X]$ for any $t \in [0,T]$, for the worst-case model $P^{\bar\theta}$. Note that here risk-minimization is under a risk-neutral measure $\bar Q$ that could also depend on the contingent claim into consideration, and not under the minimal martingale measure $\widehat{Q}$ as in the original works [FS86, Sch01]. The eventually non-speculative nature of the robust good-deal hedging strategy under (large) uncertainty offers new theoretical support for the quadratic hedging objective of risk minimization, which may be criticized for giving equal weighting for upside and downside risk. More broadly, such gives support to a common perception (see e.g. [LP00]) that speculative objectives should be avoided in hedging, in addition to more practical arguments like simplifications for marking-to-market (uses risk neutral valuation). To make the said statement precise, consider the classical BSDE

$$-dY_t = \big(- \xi_t^{0\,\mathrm{tr}} \Pi_t(Z_t) + h_t \big(\Pi_t^{\perp}(Z_t)^{\mathrm{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)\big)^{1/2} \big) dt - Z_t^{\mathrm{tr}} dW_t^0, \tag{3.70}$$

for $t \in [0,T]$ with $Y_T = X$. First we prove the following

Proposition 3.32. *Assume (3.22) and (3.47) hold and that Θ satisfies (3.65). For any $X \in L^2(P_0)$, let (Y^X, Z^X) denote the standard solution of the BSDE (3.70). Then*

$$\pi_\cdot^u(X) = Y^X \text{ and } \phi^*(X) = \Pi(Z^X) \text{ for all } X \in L^2(P_0) \tag{3.71}$$

$$\text{holds, if and only if} \qquad \mathcal{R} \cap \mathcal{M}^e(S) \neq \emptyset. \tag{3.72}$$

Proof. Let $X \in L^2(P_0)$. Recall that for Θ defined in (3.65), $\bar{\theta}$ from Theorem 3.28 does not vary with X and solves the minimization problem (3.66). Now if (3.72) holds, then there exists $\theta \in \Theta$ such that $P^\theta \in \mathcal{R} \cap \mathcal{M}^e(S) \neq \emptyset$, i.e. $\widehat{Q}^\theta = P^\theta$, and therefore $\xi^\theta = 0$. This implies that $\theta = \bar{\theta} = -\xi^0$ and hence $\xi^0 \in \Theta$. As a consequence, the generator $f = f^{\bar{\theta}}$ of the BSDE (3.55) coincides with that of the BSDE (3.70). By uniqueness of standard BSDE solutions follows $\pi_\cdot^u(X) = Y^X$. Now from Theorem 3.28 and Theorem 3.30 one obtains that $\phi^* = \bar{\phi} = \Pi(Z^X)$. Conversely, suppose that (3.71) holds. Then the generator $f = f^{\bar{\theta}}$ for the BSDE (3.55) and the one for (3.70) are equal everywhere by [CHMP02, Theorem 7.1 and Rmk. 4.1]. This implies (since $\Pi^\perp(\theta) = 0$ for all $\theta \in \Theta$) that $\xi^{\bar{\theta}} = 0$, i.e. $\widehat{Q}^{\bar{\theta}} = P^{\bar{\theta}}$, and hence $\mathcal{R} \cap \mathcal{M}^e(S) \neq \emptyset$.

□

Now we can make the previously described relation between global risk minimization and good-deal hedging under (large) uncertainty precise.

Theorem 3.33. *Let the assumptions of Proposition 3.32 and (3.72) hold. For X in $L^2(P_0)$, let (Y, Z) be the standard solution of the BSDE (3.70). Then $\pi_\cdot^u(X) = Y$ has the GKW decomposition with respect to $\sigma \cdot \widehat{W}^0$ (and $S = \mathrm{diag}(S)\sigma \cdot \widehat{W}^0$, cf. Section 3.1.1)*

$$\pi_t^u(X) = \pi_0^u(X) + \phi^* \cdot \widehat{W}_t^0 + R_t^{\phi^*}, \quad t \in [0,T], \tag{3.73}$$

with $\phi^* = \Pi(Z)$. The tracking error $R^{\phi^*}(X) = \Pi_\cdot^\perp(Z) \cdot W^{\bar{Q}}$ is a $\bar{Q}$-martingale orthogonal to $\sigma \cdot \widehat{W}$, for $\bar{Q} \in \mathcal{Q}^{ngd}$ given by $d\bar{Q}/dP_0 = \mathcal{E}\left((-\xi^0 + \bar{\eta}) \cdot W^0\right)$ with

$$\bar{\eta}_t = h_t \big(\Pi_t^\perp(Z_t)^{tr} A_t^{-1} \Pi_t^\perp(Z_t)\big)^{-1/2} A_t^{-1} \Pi_t^\perp(Z_t), \quad t \in [0,T].$$

Proof. From Proposition 3.32 we have $\phi^* = \Pi(Z)$ and $\pi^u_\cdot(X) = Y$. By the definitions of $\bar{\eta}$ and $\bar{Q}$, $Y_t = Y_0 + Z \cdot W^{\bar{Q}}_t$ holds for all $t \in [0,T]$. As a consequence, one obtains $\pi^u_t(X) = \pi^u_0(X) + \phi^* \cdot \widehat{W}^0_t + \Pi^\perp_\cdot(Z_\cdot) \cdot W^{\bar{Q}}_t, t \in [0,T]$. Thus (3.73) holds with $R^{\phi^*}(X) = \Pi^\perp_\cdot(Z) \cdot W^{\bar{Q}}$ being a $\bar{Q}$-martingale orthogonal to $S = S_0 + \int_0^\cdot \mathrm{diag}(S_t)\sigma_t dW^{\bar{Q}}_t$ since $\sigma(\Pi^\perp_\cdot(Z)) = 0$. Furthermore since $\phi^*_t \perp \Pi^\perp_t(Z_t)$, and $\phi^* \cdot \widehat{W}^0 = \phi^* \cdot W^{\bar{Q}}$, then $R^{\phi *}(X)$ is also orthogonal to $\phi^* \cdot \widehat{W}^0$ under $\bar{Q}$. Therefore (3.73) is the GKW decomposition of $\bar{\pi}^u_\cdot(X)$ under $\bar{Q}$. □

Remark 3.34. *Note that using Section 3.1.1 and $dS/S = \sigma d\widehat{W}^0$, any Galtchouk-Kunita-Watanabe (GKW) decomposition (see [Sch01]) of a continuous local Q-martingale M for Q in $\mathcal{Q}^{ngd}$ with respect to $\sigma \cdot \widehat{W}^0$ gives a GKW decomposition with respect to S and vice versa. In this sense, Theorem 3.33 shows that the robust good-deal hedging strategy ϕ^* for X coincides with the (global) risk-minimizing strategy of [FS86] (cf. [Sch01, Section 2]) with respect to a specific measure $\bar{Q} = \bar{Q}(X) \in \mathcal{Q}^{ngd}$ (instead of $\widehat{Q}^0$) under which $\pi^u_t(X)$ is equal to $E^{\bar{Q}}_t[X]$ (instead of $E^{\widehat{Q}^0}_t[X]$), $t \in [0,T]$. Note that $\bar{Q}(X)$ is not equal to $\widehat{Q}^0$ in general unless $h = 0$, in which case equality holds for any contingent claim $X \in L^2(P_0)$, or X is replicable.*

A seminal no-trade result by [DW92] shows that a utility optimizing agent abstains from taking any position in a tradeable risky asset if uncertainty is too large. In comparison, the above theorem shows that a good-deal hedger keeps dynamically trading according to the risk minimizing component $\Pi(Z)$ but ceases to comprise any speculative component. [BCCH14] demonstrate by numerical computation in an example, in a setting quite different to ours, that the relative benefit of dynamic hedging compared to static hedging could decrease if uncertainty increases. This is intuitive, since (see e.g. [Con06]) static hedges can be less exposed to model risk. Proposition 3.32 likewise addresses how increasing uncertainty affects dynamic hedging, but is different in that it offers theoretical conditions under which dynamic good-deal hedging ϕ^* ceases to comprise speculative components in order to compensate for exposures to non-spanned risk.

3.3.6 Example with closed-form solutions under model uncertainty

The usual filtration is generated by a two-dimensional Brownian motion $W^0 = (W^{0,S}, W^{0,H})^{\mathrm{tr}}$ under P_0. We consider a single traded risky asset with price S and a non-traded asset with value H modelled under P_0 for $t \in [0,T]$ by

$$dS_t = S_t \sigma^S (\xi^{0,S} dt + dW^{0,S}_t), \quad dH_t = H_t(\gamma dt + \beta(\rho dW^{0,S}_t + \sqrt{1-\rho^2} dW^{0,H}_t))$$

with $S_0, H_0 > 0$, scalars $\sigma^S, \beta > 0$, $\gamma, \xi^{0,S} \in \mathbb{R}$ and correlation coefficient $\rho \in [-1,1]$. We derive robust good-deal bounds and hedging strategies in closed-form, for European call options on the non-traded asset and for no-good-deal constraint and uncertainty modelled (as in

Section 3.3.5) using the radial sets $C^0 = \{x \in \mathbb{R}^2 : |x| \le h\}$ and $\Theta^0 = \{x \in \mathbb{R}^2 : |x| \le \delta\}$ for scalars $h, \delta \ge 0$. Here one has $\Theta = \Theta^0 \cap \operatorname{Im}\sigma = [-\delta, \delta] \times \{0\}$, for $\sigma = (\sigma^S, 0)$, and hence $\xi^\theta = (\xi^{\theta,S}, 0)^{\mathrm{tr}} := (\xi^{0,S} + \theta^S, 0)^{\mathrm{tr}} \in \operatorname{Im}\sigma$, for models P^θ with $\theta = (\theta^S, 0)^{\mathrm{tr}}$, where $\theta^S \in [-\delta, \delta]$. From (3.69), with $A = B \equiv \mathrm{Id}_{\mathbb{R}^2}$ and $r = 1$, the worst-case model $P^{\bar\theta}$ corresponds to

$$\bar\theta^S = -\xi^{0,S} I_{\{|\xi^{0,S}| \le \delta\}} - \delta \frac{\xi^{0,S}}{|\xi^{0,S}|} I_{\{|\xi^{0,S}| > \delta\}}. \tag{3.74}$$

By Theorems 3.28,3.30, the robust good-deal bound and hedging strategy for a call option $X := (H_T - K)^+$ are given by $\pi^u_\cdot(X) = Y$ and $\bar\phi(X) = \Big(Z^1 + \frac{|Z^2|}{\sqrt{h^2 - |\xi^{\bar\theta,S}|^2}} \xi^{\bar\theta,S}, 0\Big)^{\mathrm{tr}}$, for $(Y, Z := (Z^1, Z^2)^{\mathrm{tr}})$ solving the BSDE (3.55), equaling the BSDE (3.51) for $\theta = \bar\theta$:

$$-dY_t = \big(-\xi^{\bar\theta,S} Z^1_t + (h^2 - |\xi^{\bar\theta,S}|^2)^{1/2} |Z^2_t| \big) dt - Z^{\mathrm{tr}}_t dW^{\bar\theta}_t \quad \text{and} \quad Y_T = X, \tag{3.75}$$

with $W^{\bar\theta}_t := (W^{0,S}_t - \bar\theta^S t, W^{0,H}_t)^{\mathrm{tr}},\ t \in [0, T]$. Writing (3.75) under $\widehat{Q}^{\bar\theta}$ and using (3.74), arguments analogous to those in the derivation of (3.31) yield

$$\begin{aligned}
\pi^u_t(X) &= N(d_+) H_t e^{\tilde\alpha_+(T-t)} - K N(d_-)\\
&=: e^{\tilde\alpha_+(T-t)} * \text{B/S-call-price}(\text{time: } t,\ \text{spot: } H_t,\ \text{strike: } K e^{-\tilde\alpha_+(T-t)}, \text{vol: } \beta),\\
\pi^l_t(X) &= e^{\tilde\alpha_-(T-t)} * \text{B/S-call-price}(\text{time: } t,\ \text{spot: } H_t,\ \text{strike: } K e^{-\tilde\alpha_-(T-t)}, \text{vol: } \beta),
\end{aligned}$$

with

$$d_\pm := \big(\ln(H_t/K) + (\tilde\alpha_+ \pm \tfrac{1}{2}\beta^2)(T-t) \big) / (\beta\sqrt{T-t}),$$
$$\tilde\alpha_\pm := \gamma + \beta\big(-\rho\xi^{0,S} \pm \tilde h \sqrt{1-\rho^2} \big)$$

and

$$\tilde h = \tilde h(\delta) := (h^2 - |\xi^{\bar\theta,S}|^2)^{1/2} = h I_{\{|\xi^{0,S}| \le \delta\}} + \big(h^2 - \big|\xi^{0,S} - \delta\big|^2\big)^{1/2} I_{\{|\xi^{0,S}| > \delta\}}$$

. Analogously to the derivation of (3.32), note that $Z = e^{\tilde\alpha_+(T-t)} N(d_+) H_t \beta(\rho, \sqrt{1-\rho^2})^{\mathrm{tr}}$. Hence the (seller's) robust good-deal hedging strategy is obtained as

$$\bar\phi_t(X) = e^{\tilde\alpha_+(T-t)} N(d_+) H_t \beta \left(\rho + \frac{\sqrt{1-\rho^2}\, \xi^{0,S}}{\tilde h |\xi^{0,S}|} (|\xi^{0,S}| - \delta) \mathbb{1}_{\{|\xi^{0,S}| > \delta\}}, 0 \right)^{\mathrm{tr}},\ t \in [0, T].$$

For $|\xi^{0,S}| > \delta$, the speculative nature of $\bar\phi(X)$ is reflected by the presence of the second summand in the first component of $\bar\phi(X)$ above. For $|\xi^{0,S}| \le \delta$, this summand vanishes and the function $\delta \mapsto \tilde\alpha_+$ is constant on $\delta \in [|\xi^{0,S}|, \infty]$. In this case robust good-deal hedging is then globally risk-minimizing with respect to the measure $\bar Q = Q^{\bar\lambda} \in \mathrm{Q}^{\mathrm{ngd}}(P_0)$ with Girsanov kernel $\bar\lambda := \big(-\xi^{0,S}, h \big)^{\mathrm{tr}}$ and non-speculative as proved in Theorem 3.33. Note that for $\delta = \xi^{0,S} = 0$ (i.e. risk-neutral setting under $P_0 = \widehat{Q}^0$ in absence of uncertainty), we recover formulas of Section 3.2.2 for $n = 2$ and $d = 1$.

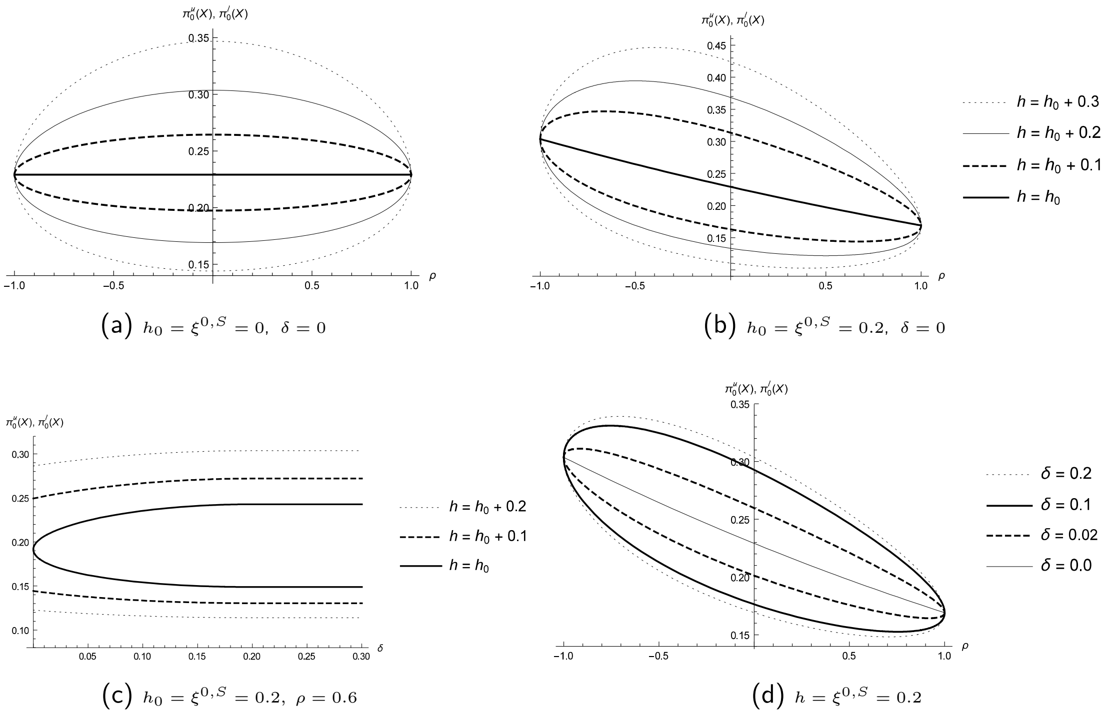
Figure 3.4 illustrates the dependence of the bounds $\pi_0^u(X)$ and $\pi_0^l(X)$ in the presence of uncertainty, on the correlation coefficient ρ, uncertainty size δ and no-good-deal constraint (optimal growth rate bound) h, and for global parameters $\gamma = 0.05$, $\beta = 0.5$, $K = 1$, $H_0 = 1$ and $T = 1$. Figures 3.4a,3.4b are plots of $\pi_0^u(X)$ and $\pi_0^l(X)$ as functions of ρ for fixed $\delta = 0$ (i.e. absence of uncertainty) and $\xi^{0,S} \in \{0, 0.2\}$, showing how the good-deal bounds vary for different values of h. Figure 3.4d contains a similar plot for fixed $h = \xi^{0,S} = 0.2$, showing how the bounds vary with ρ for different values of δ. One can observe that the maximum of $\pi_0^u(X)$ and minimum of $\pi_0^l(X)$ are attained at $\rho = 0$ only for $\xi^{0,S} = 0$ (cf. Figure 3.4a). In other words, if the market price of risk $\xi^{0,S}$ is zero (hence $P_0 = \widehat{Q}^0$), then the largest good-deal bounds are obtained when the traded and non-traded assets are uncorrelated (i.e. $\rho = 0$). On the other hand if $\xi^{0,S} > 0$ (as e.g. in Figures 3.4b,3.4d), the plots are tilted so that the maximum of $\pi_0^u(X)$ (resp. minimum of $\pi_0^l(X)$) is reached at $\rho < 0$ (resp. $\rho > 0$). For $\pi_0^u(X)$, this is explained by the fact that if the market price of risk $\xi^{0,S}$ is positive, the supremum in (3.3) is maximized by the no-good-deal measure $\bar{Q} = Q^{\bar{\lambda}} \in \mathrm{Q}^{\mathsf{ngd}}(P_0)$ with Girsanov kernel $\bar{\lambda} := \big(-\xi^{0,S}, \tilde{h}\big)^{\mathsf{tr}}$ under which the upward drift $\tilde{\alpha}_+$ of the underlying price process H is maximized, clearly at a negative correlation ρ. The explanation for $\pi_0^u(X)$ is similar, with $\tilde{\alpha}_-$ being minimal at a positive correlation, for $\xi^{0,S} > 0$. For $\xi^{0,S} < 0$ the tilt of the plots occurs in the other direction. That the good-deal bounds in Figures 3.4a,3.4b,3.4d coincide for perfect correlation $\rho = \pm 1$ is clear, because in this case derivatives X on H are attainable and admit unique no-arbitrage prices, implying $\pi_\cdot^u(X) = \pi_\cdot^l(X)$. Finally, Figure 3.4c illustrates the evolution with respect to δ of the good-deal bounds at time $t = 0$ for $\rho = 0.6$, $\xi^{0,S} = 0.2$ and different values of h, with $|\xi^{0,S}|$ chosen as the smallest value h_0 of h. One observes that for each given h, the good-deal bound curves become flat for $\delta \geq |\xi^{0,S}|$ (as predicted by Proposition 3.32), and match (i.e. $\pi_0^u(X) = \pi_0^l(X)$) for $\delta = |\xi^{0,S}| - h_0 = 0$ (as might be expected in the absence of uncertainty for a degenerate expected growth rate bound $h = |\xi^{0,S}|$).

3.4 Appendix

This appendix includes lemmas and proofs omitted from the main body of the chapter. For the convenience of the reader, some derivations are detailed as well.

Lemma 3.35. *For $d < n$, let $\sigma \in \mathbb{R}^{d\times n}$ be of full-rank, $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and $h > 0$, $Z \in \mathbb{R}^n$, $\xi \in \operatorname{Im}\sigma^{tr}$. Let $\alpha' > 0$ be a constant of ellipticity of A^{-1} and assume that $|\xi| < h\sqrt{\alpha'}$ and $A^{-1}(\operatorname{Ker}\sigma) = \operatorname{Ker}\sigma$. Then the vector $\bar{\phi} := \Pi(Z) + \big(\Pi^{\perp}(Z)^{tr}A^{-1}\Pi^{\perp}(Z)\big)^{1/2}\big(h^2 - \xi^{tr}A\xi\big)^{-1/2}A\xi$ is the unique minimizer of the function $\phi \mapsto F(\phi) := -\xi^*\phi + h\big((Z-\phi)^{tr}A^{-1}(Z-\phi)\big)^{1/2}$ on $\operatorname{Im}\sigma^{tr}$.*

Proof. Since $A^{-1}(\operatorname{Ker}\sigma) = \operatorname{Ker}\sigma$, then $\bar{\phi} \in \operatorname{Im}\sigma^{\mathrm{tr}}$. The Kuhn-Tucker optimality conditions



(a) $h_0 = \xi^{0,S} = 0,\ \delta = 0$ (b) $h_0 = \xi^{0,S} = 0.2,\ \delta = 0$

(c) $h_0 = \xi^{0,S} = 0.2,\ \rho = 0.6$ (d) $h = \xi^{0,S} = 0.2$

Figure 3.4: Dependence of $\pi_0^u(X), \pi_0^l(X)$ on $\rho,\ h$ and/or δ

are satisfied by $\bar{\phi}$. The function F is convex and differentiable at every $\phi \neq Z$, where its gradient is $\partial F(\phi) = -\xi - h\big((Z-\phi)^{\rm tr}A^{-1}(Z-\phi)\big)^{-1/2}A^{-1}(Z-\phi)$. This yields $\partial F(\bar{\phi}) = -\big(h^2 - \xi^{\rm tr}A\xi\big)^{1/2}\big(\Pi^{\perp}(Z)^{\rm tr}A^{-1}\Pi^{\perp}(Z)\big)^{-1/2}A^{-1}\Pi^{\perp}(Z)$ for $\bar{\phi} \neq Z$, using $A^{-1}(\mathrm{Ker}\ \sigma) = \mathrm{Ker}\ \sigma$. At $\phi = Z$ the subgradient is well-defined and the inclusion $E := \{x \in \mathbb{R}^n \mid |x| \leq h\sqrt{\alpha'} - |\xi|\} \subseteq \partial F(\phi)$ holds. Overall $\partial F(\bar{\phi}) = -\big(h^2 - \xi^{\rm tr}A\xi\big)^{1/2}\big(\Pi^{\perp}(Z)^{\rm tr}A^{-1}\Pi^{\perp}(Z)\big)^{-1/2}A^{-1}\left(\Pi^{\perp}(Z)\right)$ for $\bar{\phi} \neq Z$ and $0 \in E \subseteq \partial F(\bar{\phi})$ for $\bar{\phi} = Z$. In any case, $\bar{\phi}$ satisfies the Karush-Kuhn-Tucker conditions and since F is convex and the minimization constraint $\phi \in \mathrm{Im}\ \sigma^{\rm tr}$ is linear, optimality of $\bar{\phi}$ follows from the Kuhn-Tucker theorem (cf. [Roc70, Section 28]). Uniqueness of $\bar{\phi}$ is implied by the fact that F is strictly convex over $\mathrm{Im}\ \sigma^{\rm tr}$ if $\Pi^{\perp}(Z) \neq 0$ and strictly convex at $\bar{\phi}$ if $\Pi^{\perp}(Z) = 0$ since $|\xi| < h\sqrt{\alpha'}$.

□

Lemma 3.36. *Let $d < n$, $h > 0$ be constant, $Z \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ a symmetric positive definite matrix, $\sigma \in \mathbb{R}^{d\times n}$ a full (d)-rank matrix, and $\xi^0 \in \Phi :=$ Im σ^{tr}. Let $\Theta \subset \mathbb{R}^n$ be a convex-compact set, and $F : \mathbb{R}^n \times \mathbb{R}^n \ni (\phi,\theta) \mapsto \theta^{tr}(Z-\phi) - {\xi^0}^{tr}\phi + h\big((Z-\phi)^{tr}A^{-1}(Z-\phi)\big)^{1/2}$. Then the minmax identity $\inf_{\phi\in\Phi}\ \sup_{\theta\in\Theta} F(\phi,\theta) = \sup_{\theta\in\Theta}\ \inf_{\phi\in\Phi} F(\phi,\theta)$. holds.*

Proof. For all $\phi \in \mathbb{R}^n$, the function $\theta \mapsto F(\phi,\theta)$ is concave, continuous. For all $\theta \in \mathbb{R}^n$ the function $\phi \mapsto F(\phi, \theta)$ is convex and continuous. As $\Theta \subset \mathbb{R}^n$ is convex and compact, and $\Phi = \text{Im } \sigma^{\text{tr}}$ is convex and closed, a minimax theorem [ET99, Chapter VI, Proposition 2.3] applies and the minmax identity holds.

□

Proof of Lemma 3.1. Part a) is classical (see [Del06] and cf. previously given other references). As for Part b), m-stability and convexity of $\mathcal{M}^e$ follow from [Del06, Proposition 5]. Convexity of $\mathcal{Q}^{\mathrm{ngd}}$ follows from that of $\mathcal{M}^e$ and the values of C. To show m-stability of $\mathcal{Q}^{\mathrm{ngd}}$, let $Z^i = \mathcal{E}(\lambda^i \cdot W) \in \mathcal{Q}^{\mathrm{ngd}},\ i = 1, 2,\ \tau \le T$ be a stopping time and $Z = I_{[0,\tau]} Z^1_\cdot + I_{]\tau,T]} Z^1_\tau Z^2_\cdot / Z^2_\tau$. Since $\mathcal{M}^e$ is m-stable, then $Z \in \mathcal{M}^e$ and one has $Z = \mathcal{E}(\lambda \cdot W)$ for some predictable process λ. It remains to show that λ is bounded and that $\lambda \in C$. From the expression of Z, writing the densities Z, Z^1, Z^2 as ordinary exponentials by distinguishing $t \le \tau$ and $t \ge \tau$, and taking the logarithm yields $(\lambda - I_{[0,\tau]}\lambda^1 - I_{]\tau,T]}\lambda^2) \cdot W = \frac{1}{2} \int_0^\cdot \big(|\lambda_s|^2 - I_{[0,\tau]}(s) \, |\lambda^1_s|^2 - I_{]\tau,T]}(s) \, |\lambda^2_s|^2 \big) ds$. Since $\mathbb{F}$ is the augmented Brownian filtration, then $[0, \tau]$ and $]\tau, T]$ are predictable and so is $\lambda - I_{[0,\tau]}\lambda^1 - I_{]\tau,T]}\lambda^2$. Hence $\left(\lambda - I_{[0,\tau]}\lambda^1 - I_{]\tau,T]}\lambda^2\right) \cdot W$ is a continuous local martingale of finite variation and is thus equal to zero. As a consequence $\lambda = I_B \lambda^1 + I_{B^c}\lambda^2$ is bounded since λ^1, λ^2 are, and satisfies $\lambda \in C$ since C is convex-valued.

□

Proof of Theorem 3.7. Without loss of generality, we argue only for $X \ge 0$; otherwise one can use translation invariance with $X + \|X\|_\infty \ge 0$.

Part 1: Let $t \in [0, T]$ and define $\widetilde{\pi}_t(X) := \text{ess sup}_{Q \in \overline{\mathcal{Q}^{\mathrm{ngd}}}} E^Q_t[X]$. We have the inclusions $C^k_t(\omega) \subseteq C^{k+1}_t(\omega) \subseteq C_t(\omega)$ for all (t, ω) and for all $k \in \mathbb{N}$, and hence the chain of inequalities $\pi^{u,k}_t(X) \le \pi^{u,k+1}_t(X) \le \pi^u_t(X) \le \widetilde{\pi}_t(X)$ holds for $k \in \mathbb{N}$. Since the sequence $(\pi^{u,k}_t(X))_{k\in\mathbb{N}}$ is non-decreasing and $|\pi^{u,k}_t(X)| \le \|X\|_\infty$, for all k, the monotone a.s. limit $J_t := \lim_{k \nearrow \infty} \pi^{u,k}_t(X)$ is finite and $\widetilde{\pi}_t(X) \ge J_t$. It remains to show the reverse inequality, which implies $\widetilde{\pi}_t(X) = \pi^u_t(X)$, using Part 2 of Proposition 3.5 to $\pi^{u,k}_\cdot(X)$ to obtain a sequence of measures $\bar{Q}^k \in \mathcal{Q}^{\mathrm{ngd}}_k \subseteq \mathcal{Q}^{\mathrm{ngd}}$ satisfying $\pi^u_t(X) \ge E^{\bar{Q}^k}_t[X] = \pi^{u,k}_t(X) \nearrow \widetilde{\pi}_t(X)$ as $k \to \infty$. To this end, it suffices to show that J is a càdlàg Q-supermartingale for all $Q \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$ and then apply Lemma 3.6 to $\widetilde{\pi}_\cdot(X) = \pi^{u,\overline{\mathcal{Q}^{\mathrm{ngd}}}}_\cdot(X)$ since $\overline{\mathcal{Q}^{\mathrm{ngd}}}$ is also convex and m-stable (argument being analogous to that for $\mathcal{Q}^{\mathrm{ngd}}$ in Lemma 3.1) b). First notice that J is a càdlàg Q-supermartingale for any $Q \in \mathcal{Q}^{\mathrm{ngd}}$ with Girsanov kernel $\lambda^Q = \lambda$. Indeed for such measures Q, there exists $k_0 \in \mathbb{N}$ such that $\lambda \in \Lambda^k$ for all $k \ge k_0$. Since $J_t = \lim_k \pi^{u,k}_t(X)$ and $\pi^{u,k}_\cdot(X)$ is a bounded càdlàg Q-supermartingale for every $k \ge k_0$, then J is a càdlàg Q-supermartingale as the increasing limit of càdlàg Q-supermartingales of class D (cf. [Doo01, Section 2.IV.4]). Now let $Q \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$ with $\lambda^Q = \lambda = -\xi + \eta \in \Lambda$ not necessarily bounded. Then $\lambda^n := -\xi + \eta^n$ with $\eta^n := \eta I_{\{|\eta| \le n\}} \in \mathrm{Ker}\ \sigma$ forms a sequence of bounded Girsanov kernels for measures

$Q^n \in \mathcal{Q}^{\mathrm{ngd}}$ such that $\lim_{n\to\infty} \lambda^n = \lambda \; P \otimes dt -$ a.e.. By the above arguments, since ξ and X are bounded, then J is a bounded càdlàg $\widehat{Q}$-supermartingale (hence of class D) that admits a Doob-Meyer decomposition which, by the predictable representation property of $\widehat{W}$ with respect $(\widehat{Q}, \mathbb{F})$ (cf. [HWY92, Theorem 13.22]), reads $J = J_0 + Z \cdot \widehat{W} - A$, where $Z \in \mathcal{H}^2(\widehat{Q})$ and A is a non-decreasing predictable processes with $A_0 = 0$ and $A_T \in L^2(\widehat{Q})$ because $J \in \mathcal{S}^\infty$ is bounded (cf. [DM82, Inequality (15.1), Section VII.15, page 202]). One rewrites

$$J = J_0 + Z \cdot W^{Q^n} + \int_0^\cdot Z_t^{\mathrm{tr}} \eta_t^n dt - A, \text{ and} \tag{3.76}$$

$$J = J_0 + Z \cdot W^{Q} + \int_0^\cdot Z_t^{\mathrm{tr}} \eta_t dt - A. \tag{3.77}$$

Since the Girsanov kernels λ^n are bounded, J is a càdlàg supermartingale under Q^n for all n. Hence from (3.76) one has $dA_t \geq Z_t^{\mathrm{tr}} \eta_t^n dt,\ t \in [0,T]$, for all $n \in \mathbb{N}$. By dominated convergence, taking the limit as $n \to \infty$ implies $dA_t \geq Z_t^{\mathrm{tr}} \eta_t dt,\ t \in [0,T]$. Now since X is non-negative, then so is J and positivity of $-\int_0^\cdot Z_t^{\mathrm{tr}} \eta_t dt + A$ implies $Z \cdot W^Q \geq$ const from (3.77). Being bounded from below, the local Q-martingale $Z \cdot W^Q$ is therefore a Q-supermartingale. Finally because J is bounded, then $\int_0^T Z_t^{\mathrm{tr}} \eta_t dt - A_T$ is Q-integrable and thus J is a Q-supermartingale.

Part 2: For $k \in \mathbb{N}$, the process $\pi^{u,k}$ is the good-deal bound associated to the constraint correspondence C^k satisfying Assumption 3.3. Hence applying Part 2 of Proposition 3.5 with C replaced by C^k yields the result.

Part 3: For all $k \geq \|\xi\|_\infty$ holds $\widehat{Q} \in \mathcal{Q}_k^{\mathrm{ngd}} \subset \mathcal{Q}^{\mathrm{ngd}}$. Hence by Lemma 3.6, $\pi_\cdot^u(X)$ and $\pi_\cdot^{u,k}(X)$ are bounded càdlàg $\widehat{Q}$-supermartingales, admitting Doob-Meyer decompositions under $\widehat{Q}$ as in (3.12), and by Part 2 A^k satisfies (3.13). By arguments similar to those for Part 1 follows $Z, Z^k \in \mathcal{H}^2(\widehat{Q})$ and $A, A^k \in L^2(\widehat{Q})$.

Part 4: From Part 3, that A_u^k converges to A_u weakly in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$ for all $u \leq T$ follows from [DM82, Theorem VII.18 and subsequent remarks]. These apply since the sequence $(\pi_\cdot^{u,k}(X))_{k \geq \|\xi\|_\infty}$ is uniformly bounded by $\|X\|_\infty$, and hence Part 1 and dominated convergence imply that $\pi_u^{u,k}(X)$ converges to $\pi_u^u(X)$ in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$, for all $u \in [0,T]$. Furthermore the convergences of $(\pi_u^{u,k}(X))_k$ and $(A_u^k)_k$ imply that $Z^k \cdot \widehat{W}_u \to Z \cdot \widehat{W}_u$ weakly in $L^2(\Omega, \widehat{Q}, \mathcal{F}_u)$ for all $u \in [0,T]$. By the predictable representation property and Itô's isometry, follows $Z^k \to Z$ weakly in $L^2(\Omega \times [0,u], \widehat{Q} \otimes dt)$ for any u.

□

Proof of Theorem 3.9. By Theorem 3.7, $\pi_\cdot^u(X)$ admits under $\widehat{Q}$ the Doob-Meyer decomposition $\pi_\cdot^u(X) = \pi_0^u(X) + Z \cdot \widehat{W} - A = \pi_0^u(X) + Z \cdot W + \int_0^\cdot \xi_t^{\mathrm{tr}} Z_t dt - A$, where $Z \in \mathcal{H}^2(\widehat{Q})$ and A is a non-decreasing predictable process with $A_0 = 0$. Alternatively one rewrites $-d\pi_t^u(X) = g_t(Z_t)dt - Z_t^{\mathrm{tr}} dW_t + dK_t$, with $K := A - \int_0^\cdot \xi_t^{\mathrm{tr}} Z_t dt - \int_0^\cdot \operatorname{ess\,sup}_{\lambda_t \in \Lambda_t} \lambda_t^{\mathrm{tr}} Z_t \; dt$ being finite-valued and predictable. For $(\pi_\cdot^u(X), Z, K)$ to be a supersolution to the BSDE with parameters (g, X) it suffices to show that K is non-decreasing. For any $\lambda = -\xi + \eta \in \Lambda$, one can construct the sequence of $\lambda^n = -\xi + \eta^n \in \Lambda$ Girsanov kernels of measures $Q^n \in \mathcal{Q}^{\mathrm{ngd}}$ with $\eta^n = \eta I_{\{|\eta| \le n\}}$ such that $\lambda^n \to \lambda$ $P \otimes dt$-a.s. as $n \to \infty$. For each Q^n it holds $\pi^u_\cdot(X) = \pi^u_0(X) + Z \cdot W^{Q^n} + \int_0^\cdot Z_t{}^{\mathrm{tr}} \eta^n_t dt - A$. Since $\pi^u_\cdot(X)$ is a bounded Q^n-supermartingale, then $dA_t - \xi^{\mathrm{tr}}_t Z_t dt \ge Z^{\mathrm{tr}}_t \lambda^n_t dt$, for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using dominated convergence one obtains $dA_t - \xi^{\mathrm{tr}}_t Z_t dt \ge Z^{\mathrm{tr}}_t \lambda_t dt$. Now taking the essential supremum over all $\lambda \in \Lambda$ yields $dK_t \ge 0$.

To show that the supersolution $(\pi^u_\cdot(X), Z, K)$ is minimal, it suffices (by Lemma 3.6) to show that the Y-component of any other supersolution is a càdlàg Q-supermartingale for every $Q \in \mathcal{Q}^{\mathrm{ngd}}$. Let $(\bar{Y}, \bar{Z}, \bar{K})$ be a supersolution of the BSDE with parameters (g, X), with $\bar{Y} \in \mathcal{S}^\infty$. By change of measure, the dynamics of $\bar{Y}$ under some measure $Q \in \mathcal{Q}^{\mathrm{ngd}}$ with Girsanov kernel $\lambda^Q \in \Lambda$ is

$$-d\bar{Y}_t = \Big(\operatorname*{ess\,sup}_{\lambda_t \in \Lambda_t} \lambda^{\mathrm{tr}}_t \bar{Z}_t - \bar{Z}^{\mathrm{tr}}_t \lambda^Q_t\Big) dt - \bar{Z}^{\mathrm{tr}}_t dW^Q_t + d\bar{K}_t, \quad t \in [0, T]. \tag{3.78}$$

Since $\bar{K}$ is non-decreasing, it holds that

$$d\bar{K}_t + \Big(\operatorname*{ess\,sup}_{\lambda_t \in \Lambda_t} \lambda^{\mathrm{tr}}_t \bar{Z}_t - \bar{Z}^{\mathrm{tr}}_t \lambda^Q_t\Big) dt \ge 0, \quad t \in [0, T]. \tag{3.79}$$

From (3.79), (3.78) and boundedness of $\bar{Y}$, the local martingale $\bar{Z} \cdot W^Q$ is bounded from below, hence is a supermartingale. Again since $\bar{Y} \in \mathcal{S}^\infty$, then the integral of (3.79) in $[0, T]$ is Q-integrable and therefore $\bar{Y}$ is a Q-supermartingale.

□

Proof of Corollary 3.10. By m-stability and convexity of $\overline{\mathcal{Q}^{\mathrm{ngd}}}$, Lemma 3.6 and Part 1. of Theorem 3.7 imply that $\pi^u_\cdot(X)$ is a càdlàg $\bar{Q}$-supermartingale with terminal value X since $\bar{Q} \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$. We have

$$E^{\bar{Q}}[X] = \pi^u_0(X) \ge E^{\bar{Q}}[\pi^u_t(X)] \ge E^{\bar{Q}}[E^{\bar{Q}}_t[X]] = E^{\bar{Q}}[X], \ t \le T.$$

Hence $\pi^u_\cdot(X)$ is a $\bar{Q}$-martingale. Let $Z \in \mathcal{H}^2(\widehat{Q})$ from Theorem 3.9 with

$$K := A - \int_0^\cdot \xi^{\mathrm{tr}}_t Z_t dt - \int_0^\cdot \operatorname*{ess\,sup}_{\lambda_t \in \Lambda_t} \lambda^{\mathrm{tr}}_t Z_t dt$$

such that $(\pi^u_\cdot(X), Z, K)$ is the minimal supersolution to the BSDE with parameters (g, X). One writes $\pi^u_\cdot(X)$ under $\bar{Q}$ as $\pi^u_\cdot(X) = \pi^u_0(X) + Z \cdot W^{\bar{Q}} + \int_0^\cdot Z^{\mathrm{tr}}_t(\bar{\lambda}_t + \xi_t) dt - A$. Since $\pi^u_\cdot(X)$ is a bounded $\bar{Q}$-martingale, then $A - \int_0^\cdot Z^{\mathrm{tr}}_t(\bar{\lambda}_t + \xi_t) dt = 0$. Therefore since $\bar{\lambda} \in \Lambda$, one obtains $0 \le K \le A - \int_0^\cdot Z^{\mathrm{tr}}_t(\bar{\lambda}_t + \xi_t) dt = 0$. Thus $K = 0$ and hence $(\pi^u_\cdot(X), Z)$ is a BSDE solution. Any solution being a supersolution, minimality follows from Theorem 3.9. Finally since the $\bar{Q}$-martingale $\pi^u_\cdot(X)$ satisfies $-d\pi^u_t(X) = \bar{\lambda}^{\mathrm{tr}}_t Z_t dt - Z^{\mathrm{tr}}_t dW_t$, with $\pi^u_T(X) = X$, then ess $\sup_{\lambda_t \in \Lambda_t} \lambda^{\mathrm{tr}}_t Z_t = \bar{\lambda}^{\mathrm{tr}}_t Z_t$ holds.

□

Derivation of (3.33). Consider a European option $X = G(\tilde{H}_T, \tilde{S}_T) \in L^2$ for a payoff function $(0,\infty)^2 \ni (x,y) \mapsto G(x,y) \in \mathbb{R}$ being measurable, non-decreasing in x and at most of polynomial growth in $x^{\pm 1}$, i.e. $|G(x,y)| \le k(1 + x^n + x^{-n})$ for all $(x,y) \in (0,\infty)^2$, for some $k > 0$ and $n \in \mathbb{N}$. Again following the arguments of the proof in the example of an option on $\tilde{H}$, one can show that

$$\bar{\lambda} = h\Big(\sum_{i=d+1}^{n} \tilde{\beta}_i^2/a_i\Big)^{-1/2}(0,\ldots,0,\tilde{\beta}_{d+1}/a_{d+1},\ldots,\tilde{\beta}_n/a_n)^{\mathrm{tr}}$$

and $\pi^u_t(X) = E^{\bar{Q}}_t[G(\tilde{H}_T, \tilde{S}_T)] = u(t, \tilde{H}_t, \tilde{S}_t)$ for $u \in \mathcal{C}((0,\infty) \times (0,\infty)^2)$ with $\partial_x u \ge 0$. Moreover one obtains for all $t \in [0,T]$ that

$$\begin{cases} Z^i_t = \tilde{\beta}_i \tilde{H}_t \partial_x u(t, \tilde{H}_t, \tilde{S}_t) + \tilde{\sigma}_i \tilde{S}_t \partial_y u(t, \tilde{H}_t, \tilde{S}_t), & \text{for } i \le d \text{ and} \\ Z^i_t = \tilde{\beta}_i \tilde{H}_t \partial_x u(t, \tilde{H}_t, \tilde{S}_t), & \text{for } i \ge d+1 \end{cases}$$

For the specific case $G(x,y) := (x-y)^+$, one has $X = (\tilde{H}_T - \tilde{S}_T)^+ \in L^2$. Denoting $L_t := \tilde{H}_t/\tilde{S}_t,\ t \in [0,T]$, gives $X = \tilde{S}_T(L_T - 1)^+$. A change of numéraire $d\tilde{Q}/d\bar{Q}|_{\mathcal{F}_t} = e^{-\tilde{\mu}t}\tilde{S}_t/\tilde{S}_0$ yields $\pi^u_t(X) = e^{\tilde{\mu}(T-t)}\tilde{S}_t E^{\tilde{Q}}_t[(L_T - 1)^+]$. Now $L_t = L_0 e^{(\alpha_+ - \tilde{\mu})t} \exp\big(\tilde{\beta}^{\mathrm{tr}}\tilde{W}_t - \tilde{\sigma}^{\mathrm{tr}}\tilde{W}^S_t - \frac{1}{2}\delta^2 t\big)$, with $\alpha_\pm := \tilde{\gamma} \pm h\big(\sum_{i=d+1}^n \tilde{\beta}_i^2/a_i\big)^{1/2}$, $\delta := \big(|\tilde{\beta}|^2 + |\tilde{\sigma}|^2 - 2\sum_{i=1}^d \tilde{\sigma}_i\tilde{\beta}_i\big)^{1/2}$, and $\tilde{W}$ an n-dimensional $\tilde{Q}$-Brownian motion. Now the formula (3.33) follows from the classical Margrabe formula for exchange options.

□

Derivation of (3.35),(3.37). The stochastic exponential $\mathcal{E}\big((\varepsilon/\sqrt{\nu}) \cdot W^\nu\big)$ is a uniformly integrable martingale which defines a measure $\bar{Q} \in \overline{\mathcal{Q}^{\mathrm{ngd}}} \supseteq \mathcal{Q}^{\mathrm{ngd}}$ (see (3.4) for definition of $\overline{\mathcal{Q}^{\mathrm{ngd}}} \subset \mathcal{M}^e$) with Girsanov kernel $\bar{\lambda} := \varepsilon/\sqrt{\nu}$, i.e. $d\bar{Q}/dP = \mathcal{E}\big((\varepsilon/\sqrt{\nu}) \cdot W^\nu\big)$. Indeed, applying [CFY05, Theorem 2.4 and Section 6] one gets that $\mathcal{E}\big((\varepsilon/\sqrt{\nu}) \cdot W^\nu\big)$ and $S = S_0\mathcal{E}\left(\sqrt{\nu} \cdot W^S\right)$ are uniformly integrable P- respectively Q-martingales. The variance process ν under $\bar{Q}$ is again a CIR process with parameters $(\bar{a}, b, \beta, \rho)$ where $\bar{a} := a + \beta\varepsilon\sqrt{1-\rho^2} > a$ and the Feller condition $\beta^2 \le 2\bar{a}$ still holds. For a put option $X = (K - S_T)^+ \in L^\infty$, $\bar{Y}_t := E^{\bar{Q}}_t[X]$ are given by the Heston formula (cf. [Hes93]), applied under $\bar{Q}$ (instead of P). Since the Heston price is non-decreasing in the mean reversion level of the variance process ([OA11, Proposition 5.3.1]) one expects that $\pi^u_t(X) = \bar{Y}_t = E^{\bar{Q}}_t[X]$. Let us make this precise. For $Q \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$ with Girsanov kernel λ satisfying $|\lambda| \le \varepsilon/\sqrt{\nu}$, one has $Y^Q_T = \bar{Y}_T = X$ with $Y^Q_t = E^Q_t[X]$. Using

Feynman-Kac, $\bar{Y}_t = u(t, S_t, \nu_t)$ for a function $u \in \mathcal{C}^{1,2,2}([0,T] \times \mathbb{R}^+ \times \mathbb{R}^+)$ with $\frac{\partial u}{\partial \nu} \geq 0$ (see [OA11, Theorem 5.3.1, Corollary 5.3.1]). By Itô's formula and change of measure follows

$$\begin{aligned} d\bar{Y}_t =& \beta\sqrt{1-\rho^2}\sqrt{\nu_t}(\lambda_t - \frac{\varepsilon}{\sqrt{\nu_t}})\frac{\partial u}{\partial \nu}(t, S_t, \nu_t)dt + \beta\sqrt{1-\rho^2}\sqrt{\nu_t}\frac{\partial u}{\partial \nu}(t, S_t, \nu_t)dW_t^{Q,\nu} \\ &+ \Big(S_t\sqrt{\nu_t}\frac{\partial u}{\partial S}(t, S_t, \nu_t) + \beta\rho\sqrt{\nu_t}\frac{\partial u}{\partial \nu}(t, S_t, \nu_t)\Big)dW_t^S, \quad t \in [0,T]. \end{aligned} \tag{3.80}$$

Since X is bounded, then $\bar{Y}$ is in $\mathcal{S}^\infty(Q)$ and a Q-supermartingale by (3.80) . Hence $Y_t^Q \leq \bar{Y}_t$ for all $Q \in \overline{\mathcal{Q}^{\mathrm{ngd}}}$, which by Part 1. of Theorem 3.7 implies the claim and thus we obtain the Heston type formula (3.35).

Since $\bar{Q} \in \overline{\mathcal{Q}^{\mathsf{ngd}}}$ and $\pi_0^u(X) = E^{\bar{Q}}[X]$ with $X \in L^\infty$, Corollary 3.10 implies that the good-deal bound is the Y-component of the minimal solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty \times \mathcal{H}^2$ (note $P = \widehat{Q}$) of the BSDE (3.36) with generator $g_t(z) = \bar{\lambda}_t z^2 = \varepsilon z^2/\sqrt{\nu_t}$, for $z = (z^1, z^2)$, and terminal condition X. Now consider the strategy

$$\bar{\phi}_t = \bar{Z}_t^1 = S_t\sqrt{\nu_t}\frac{\partial u}{\partial S}(t, S_t, \nu_t) + \beta\rho\sqrt{\nu_t}\frac{\partial u}{\partial \nu}(t, S_t, \nu_t) = S_t\sqrt{\nu_t}\ \Delta_t + \frac{\beta\rho}{2}\mathcal{V}_t.$$

Clearly $\bar{\phi}$ is in the set $\Phi = \mathcal{H}^2(\mathbb{R})$ of permitted trading strategies since $\bar{Z} \in \mathcal{H}^2(\mathbb{R}^2)$. Recall that $\mathcal{P}^{\mathsf{ngd}}$ consists of $dQ/dP = \mathcal{E}\big((\lambda^S, \lambda^\nu) \cdot W\big)$ such that $\big|(\lambda^S, \lambda^\nu)\big| \leq \varepsilon/\sqrt{\nu}$ with (λ^S, λ^ν) being bounded. For $Q \in \mathcal{P}^{\mathsf{ngd}}$, any wealth process $\phi \cdot W^S$, $\phi \in \Phi$, is thus in $\mathcal{S}^1(Q)$. As $\mathcal{Q}^{\mathsf{ngd}} \subseteq \mathcal{P}^{\mathsf{ngd}}$ holds, clearly $\pi_t^u(X) \leq \rho_t(X - \int_t^T \phi_s dW_s^S)$ for any strategy $\phi \in \Phi$. To prove that $\bar{\phi}$ is a good-deal hedging strategy, we show the reverse inequality $\pi_t^u(X) \geq E_t^Q\big[X - \int_t^T \bar{\phi}_s dW_s^S\big]$ for all $Q \in \mathcal{P}^{\mathsf{ngd}}$. Let $Q \in \mathcal{P}^{\mathsf{ngd}}$ with Girsanov kernel (λ^S, λ^ν). Like in (3.80), we obtain for any stopping time τ that

$$\begin{aligned} \bar{Y}_{\tau\wedge T} - \int_{\tau\wedge t}^{\tau\wedge T} \bar{\phi}_s dW_s^S = \bar{Y}_{\tau\wedge t} &+ \int_{\tau\wedge t}^{\tau\wedge T} \beta\sqrt{1-\rho^2}\sqrt{\nu_s}(\lambda_s^\nu - \frac{\varepsilon}{\sqrt{\nu_s}})\frac{\partial u}{\partial \nu}(s, S_s, \nu_s)ds \\ &+ L_{\tau\wedge T} - L_{\tau\wedge t}, \end{aligned} \tag{3.81}$$

for the local Q-martingale $L := \int_0^\cdot \beta\sqrt{1-\rho^2}\sqrt{\nu_s}\frac{\partial u}{\partial \nu}(s, S_s, \nu_s)dW_s^{Q,\nu}$. By the inequalities $\frac{\partial u}{\partial \nu} \geq 0$ and $|\lambda^\nu| \leq \varepsilon/\sqrt{\nu}$ follows that $\bar{Y}_{\tau\wedge T} - \int_{\tau\wedge t}^{\tau\wedge T} \bar{\phi}_s dW_s^S$ is less than $\bar{Y}_{\tau\wedge t} + L_{\tau\wedge T} - L_{\tau\wedge t}$. Localizing L along a sequence of stopping times $\tau_n \uparrow \infty$ and taking conditional Q-expectations yields $E_t^Q[\bar{Y}_{\tau_n\wedge T} - \int_{\tau_n\wedge t}^{\tau_n\wedge T} \bar{\phi}_s dW_s^S] \leq \bar{Y}_{\tau_n\wedge t}$. Using $X \in L^\infty$ and $\bar{\phi} \cdot W^S \in S^1(Q)$, the claim then follows by dominated convergence. Hence (3.37) holds for $\mathcal{V}_t := \frac{\partial u}{\partial \sigma}(t, S_t, \nu_t) = 2\sigma_t\frac{\partial u}{\partial \nu}(t, S_t, \nu_t)$ and volatility $\sigma_t = \sqrt{\nu_t}$.

□

Proof of Lemma 3.26. Part 1: We use (3.48) to show that the set $\bigcup_{\theta\in\Theta} \mathcal{P}^{\mathrm{ngd}}(P^\theta)$ is m-stable and convex. Let $\kappa \in [0,1]$, $\tau \leq T$ be a stopping time and $Z^i = \mathcal{E}(\lambda^i \cdot W^0)$, with λ^i selection of $C^0 + \theta^i,\ \theta^i \in \Theta,\ i = 1, 2$. From the proof of the second part of Lemma 3.1 the process $Z := I_{[0,\tau]} Z^1 + I_{]\tau,T]} Z^2 Z^1_\tau / Z^2_\tau$ satisfies $Z = \mathcal{E}(\lambda \cdot W^0)$ with $\lambda = I_{[0,\tau]}\lambda^1 + I_{]\tau,T]}\lambda^2$. By convexity of the values of C^0 it follows that $\lambda \in C^0 + \theta$ for $\theta := I_{[0,\tau]}\theta^1 + I_{]\tau,T]}\theta^2 \in \Theta$ by convexity of the values of Θ. Hence Z is in $\mathcal{P}^{\text{ngd}}(P^\theta)$, and therefore $\bigcup_{\theta\in\Theta} \mathcal{P}^{\text{ngd}}(P^\theta)$ is m-stable. To show convexity, consider the density process $\tilde{Z} = \kappa Z^1 + (1-\kappa) Z^2$. Then $\tilde{Z} = \mathcal{E}(\tilde{\lambda} \cdot W^0)$ with $\tilde{\lambda} = \frac{\kappa Z^1}{\kappa Z^1 + (1-\kappa) Z^2}\lambda^1 + \frac{(1-\kappa) Z^2}{\kappa Z^1 + (1-\kappa) Z^2}\lambda^2$. Again by convexity of the values of C^0, $\tilde{\lambda} \in C^0 + \tilde{\theta}$, for $\tilde{\theta} := \frac{\kappa Z^1}{\kappa Z^1 + (1-\kappa) Z^2}\theta^1 + \frac{(1-\kappa) Z^2}{\kappa Z^1 + (1-\kappa) Z^2}\theta^2 \in \Theta$ since Θ is convex-valued. Concerning Part 2: M-stability and convexity of $\bigcup_{\theta\in\Theta} \mathcal{Q}^{\text{ngd}}(P^\theta)$ follow from that of $\mathcal{M}^e$, Part 1, and $\bigcup_{\theta\in\Theta} \mathcal{Q}^{\text{ngd}}(P^\theta) = \left(\bigcup_{\theta\in\Theta} \mathcal{P}^{\text{ngd}}(P^\theta)\right) \bigcap \mathcal{M}^e$.

$\square$

4. Hedging under good-deal bounds and volatility uncertainty: a 2BSDE approach

In this chapter, we study good-deal bounds defined from a bound on the instantaneous Sharpe ratios in the economy and a notion of robust hedging (as in Chapter 3) in the presence of volatility uncertainty. We describe worst-case good-deal bounds and robust hedging strategies in terms of solutions to 2BSDEs. In Section 4.1 we clarify the canonical setup incorporating volatility uncertainty and provide some preliminary results about 2BSDEs. Then in Section 4.2 we describe a model of the financial market under volatility uncertainty, together with a parametrization of the no-good-deal restriction in this model. Section 4.3 is devoted to the main results of the chapter, namely a 2BSDE characterization of good-deal bounds and associated hedging strategies and the fact that the latter are at least mean-self-financing uniformly over all priors (robustness). It includes in addition an example for European put options on non-tradeable assets in a Black-Scholes model with uncertain volatility, where worst-case valuations can be computed explicitly from a Black-Scholes' type formula under a worst-case prior. Robust good-deal hedging strategies are also obtained in closed-form in this example, and it is shown that they are in general not super-replicating under volatility uncertainty.

4.1 Mathematical framework and preliminaries

We consider a canonical setting with filtered probability space $(\Omega, \mathcal{F}, P^0, \mathbb{F})$. Here $\Omega := \{\omega \in \mathcal{C}([0,T], \mathbb{R}^n) : \omega(0) = 0\}$ denotes the space of continuous paths starting at 0 and equipped with the norm $\|\omega\|_\infty := \sup_{t\in[0,T]} |\omega(t)|$. The canonical process B is defined by $B_t(\omega) := \omega(t)$, for $\omega \in \Omega$ and its law is P^0, the Wiener measure. The underlying filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is generated by B and $\mathbb{F}^+ = (\mathcal{F}_t^+)_{t\in[0,T]}$ denotes its right-limit, with $\mathcal{F}_t^+ = \mathcal{F}_{t+}$. For a probability measure Q, the conditional expectation given $\mathcal{F}_t$ will be denoted by $E_t^Q[\cdot]$.

4.1.1 The local martingale measures

A probability measure P is called a local martingale measure if B is a local martingale with respect to $(\mathbb{F}, P)$. From [Kar95] (see also [Föl81]), it follows that there exists a $\mathbb{F}$-progressively measurable process denoted by $\int_0^\cdot B_s^{\mathrm{tr}} dB_s$ which coincides with the P-Itô integrals ${}^{(P)}\!\int_0^\cdot B_s dB_s^{\mathrm{tr}}$ P-a.s. for all local martingale measures P. In particular, this yields path-wise definition of the quadratic variation $\langle B\rangle$ of B as $\langle B\rangle := BB^{\mathrm{tr}} - 2\int_0^\cdot B_s dB_s^{\mathrm{tr}}$ and of its density $\widehat{a}$ with respect to the Lebesgue measure dt as

$$\widehat{a}_t(\omega) := \limsup_{\epsilon \searrow 0} \frac{\langle B\rangle_t(\omega) - \langle B\rangle_{t-\epsilon}(\omega)}{\epsilon},$$

We denote by $\overline{\mathcal{P}}_W$ the set of all local martingale measures P for which $\widehat{a}$ is well-defined and takes values P-almost surely in the space $\mathbb{S}_n^{>0} \subset \mathbb{R}^{n\times n}$ of positive definite symmetric $n\times n$-matrices. As mentioned in [STZ11], the measures in $\overline{\mathcal{P}}_W$ can be typically mutually singular. In particular, there is no dominating measure in $\overline{\mathcal{P}}_W$ and this can be illustrated by the following example of [STZ11]:

Example 4.1. *For $n=1$, $P=P^0$, $P'=P^0\circ(\sqrt{2}B)^{-1}$, $A=\{\langle B\rangle_t = t,$ for all $t\in[0,T]\}$ and $A'=\{\langle B\rangle_t = 2t,$ for all $t\in[0,T]\}$, it holds $P,P'\in\overline{\mathcal{P}}_W$, $P(A)=P'(A')=1$ and $P(A')=P'(A)=0$. Hence $P\perp P'$.*

Note that for any $P\in\overline{\mathcal{P}}_W$, the process W^P defined by $W_t^P := {}^{(P)}\!\!\int_0^t \widehat{a}_t^{-\frac{1}{2}} dB_t,\ t\in[0,T]$ is a Brownian motion under P (by Lévy characterization and since $\langle W^P\rangle_t^P = t,\ P$-a.s.). Similarly to [STZ12], we will use the so-called *strong formulation of volatility uncertainty* according to which we consider only on the local martingale measures induced by the laws of solutions to SDEs $dX_t = a_t^{1/2}(X)dB_t,\ P$-a.s.. More precisely, uncertainty will be considered only over the subclass $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$P^\alpha := P^0\circ(X^\alpha)^{-1}, \quad \text{where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, \quad P^0\text{-a.s. } t\in[0,T],$$

with $\mathbb{S}_n^{>0}-$valued $\mathbb{F}$-progressively measurable diffusion coefficient α satisfying $\int_0^T|\alpha_t|dt < \infty,\ P^0$-a.s.. The subscript S in $\overline{\mathcal{P}}_S$ stands for "strong" as in strong formulation, as opposed to W in $\overline{\mathcal{P}}_W$ which stands for "weak". The consequence of restricting oneself to the subclass $\overline{\mathcal{P}}_S$ is the aggregation property it possesses in the sense that the following lemma (see [STZ11, Lemma 8.1, Lemma 8.2]) holds.

Lemma 4.2. *For $P\in\overline{\mathcal{P}}_W$, let $\overline{\mathbb{F}}^P$ and $\overline{\mathbb{F}^{W^P}}^P$ denote respectively the P-augmentations of the filtrations $\mathbb{F}$ and $\mathbb{F}^{W^P}$. Then $\overline{\mathcal{P}}_S = \{P\in\overline{\mathcal{P}}_W:\ \overline{\mathbb{F}}^P = \overline{\mathbb{F}^{W^P}}^P\}$, and B has the martingale representation property simultaneously with respect to all $P\in\overline{\mathcal{P}}_S$. In addition, every $P\in\overline{\mathcal{P}}_S$ satisfies the Blumenthal Zero-One law.*

Remark 4.3. *1. For any $P^\alpha\in\overline{\mathcal{P}}_S$ one has $P^\alpha\circ B^{-1} = P^0\circ(X^\alpha)^{-1}$, i.e. the distribution of B under P^α coincides with the distribution of X^α under P^0. In particular with the filtration characterization of $\overline{\mathcal{P}}_S$ in Lemma 4.2, this implies that the density of the quadratic variation of B under P^α is equal to $\widehat{a}(B)=\alpha\circ\beta_\alpha(B),\ P^\alpha\otimes dt$-a.s., for some $\mathbb{F}$-progressively measurable map $\beta_\alpha:[0,T]\times\Omega\to\mathbb{R}^n$ (see [STZ13, Lemma 2.2])*

2. *Note that for any $P \in \overline{\mathcal{P}}_S$, the Blumenthal Zero-One law in Lemma 4.2 implies $E_t^P[X] = E^P[X|\mathcal{F}_t^+]$ P-a.s. for any $X \in L^1(P)$, $t \in [0,T]$. In particular, any $\mathcal{F}_t^+$-measurable random variable has a $\mathcal{F}_t$-measurable P-version.*

We will work with an even more restricted set of local martingale measures by considering for fixed $\underline{a}, \overline{a} \in \mathbb{S}_n^{>0}$, the subclass $\mathcal{P}_H$ of $\overline{\mathcal{P}}_S$ defined by

$$\mathcal{P}_H = \Big\{ P \in \overline{\mathcal{P}}_S : \ \underline{a} \le \widehat{a} \le \overline{a}, \ P \otimes dt\text{-a.e.} \Big\}. \tag{4.1}$$

Remark 4.4. *The definition of $\mathcal{P}_H$ is slightly different from the one for $\mathcal{P}_H^\kappa$ $(1 < \kappa \le 2)$ given in [STZ12, Definition 2.6] as*

$$\begin{aligned} \mathcal{P}_H^\kappa = \Big\{ P \in \overline{\mathcal{P}}_S : \ \exists \underline{a}_P, \overline{a}_P \in \mathbb{S}_n^{>0} \text{ s.t. } \underline{a}_P \le \widehat{a} \le \overline{a}_P, \ P \otimes dt\text{-a.e.}, \\ E^P\Big[\Big(\int_0^T |F_t(0,0,\widehat{a}_t)|^\kappa dt \Big)^{\frac{2}{\kappa}} \Big] < \infty \Big\} \end{aligned} \tag{4.2}$$

Indeed, the main difference is that here $\underline{a}, \overline{a}$ do not depend on the probability measures (in fact they are constant matrices) and are exogenously fixed. In addition, the condition $E^P[(\int_0^T |F_t(0,0,\widehat{a}_t)|^\kappa dt)^{\frac{2}{\kappa}}] < \infty$ for all $P \in \mathcal{P}_H^\kappa$ has been omitted. Note that for a function H such that the convex-conjugate F as defined later in (4.3) satisfies $F_\cdot(0,0,\cdot)$ is bounded and $D_{F_\cdot(0,0,\cdot)} \subseteq [\underline{a}, \overline{a}]$, i.e. if $|F_\cdot(0,0,a)| < \infty$ then $\underline{a} \le a \le \overline{a}$, one has $\mathcal{P}_H = \mathcal{P}_H^\kappa$. In this case we simply adopt the notation of [STZ12] with the subscript H in $\mathcal{P}_H$, although the current definition of $\mathcal{P}_H$ does not involve the function H. In fact, every function F that we consider later will automatically satisfy $D_{F_\cdot(0,0,\cdot)} \subseteq [\underline{a}, \overline{a}]$ and $\sup_{P \in \mathcal{P}_H} E^P[\int_0^T |F_t(0,0,\widehat{a}_t)|^2 dt] < \infty$ (see part 2 of Remark 4.7) so that actually $\mathcal{P}_H = \mathcal{P}_H^2$.

From now on we assume that

$$\mathcal{P}_H \text{ is non-empty.}$$

Analogous to [STZ12], we will use the language of quasi-sure analysis as it appears in the framework of capacities of [DM06] as follows.

Definition 4.5. *A property is said to hold $\mathcal{Q}$-quasi-surely ($\mathcal{Q}$-q.s. for short) for a family $\mathcal{Q}$ of probability measures if it holds P-almost-surely for all $P \in \mathcal{Q}$.*

Unless explicitly stated otherwise, inequalities between random variables will be understood in the $\mathcal{P}_H$-quasi-sure sense, while inequalities between $\mathbb{F}^+$-progressively measurable processes will be in the $\mathcal{P}_H \otimes dt$-quasi-sure sense, where $\mathcal{P}_H \otimes dt := \{P \otimes dt, \ P \in \mathcal{P}_H\}$.

4.1.2 Spaces and norms

For existence and uniqueness of solutions to 2BSDEs, we define as in [STZ12] the spaces and norms of interest. Some of these are already known from classical BSDE theory, and have been modified here to account for the mutual singularity of measures in $\mathcal{P}_H$.

- L^2_H denotes the space of $\mathcal{F}_T-$measurable $\mathbb{R}$-valued random variable X with

$$\|X\|^2_{L^2_H} := \sup_{P\in\mathcal{P}_H} E^P[\,|X|^2\,] < \infty,$$

- $\mathbb{H}^2_H$ the space of $\mathbb{F}^+-$progressively measurable $\mathbb{R}^n-$valued processes Z with

$$\|Z\|^2_{\mathbb{H}^2_H} := \sup_{P\in\mathcal{P}_H} E^P[\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt] < \infty,$$

- $\mathbb{D}^2_H$ the space of $\mathbb{F}^+-$progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H\text{-q.s. càdlàg paths, and } \|Y\|_{\mathbb{D}^2_H} := \|\sup_{t\in[0,T]} |Y_t|\|_{L^2_H} < \infty,$$

- $\mathbb{L}^2_H$ the space of $X \in L^2_H$ such that

$$\|X\|^2_{\mathbb{L}^2_H} := \sup_{P\in\mathcal{P}_H} E^P[\operatorname*{ess\,sup}^P_{t\in[0,T]} E_t^{H,P}[|X|^2]] < \infty,$$

with $E_t^{H,P}[X] := \operatorname*{ess\,sup}^P_{P'\in\mathcal{P}_H(t^+,P)} E_t^{P'}[X]$, and $\mathcal{P}_H(t^+,P) := \left\{P' \in \mathcal{P}_H : P' = P \text{ on } \mathcal{F}_t^+\right\}$.

- $\mathrm{UC_b}(\Omega)$ denotes the collection of all bounded uniformly continuous maps $X : \Omega \to \mathbb{R}$ with respect to the norm $\|\cdot\|_\infty$, and
- $\mathcal{L}^2_H$ the closure of $\mathrm{UC_b}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathbb{L}^2_H}$.

4.1.3 Second order backward stochastic differential equations

In this section, we first present an existence and uniqueness theorem for solutions to 2BSDEs and a representation of these solutions in terms of those of standard BSDEs. Our exposition follows [STZ12]. These results will be used later to characterize the good-deal bounds in the presence of volatility uncertainty as solutions to 2BSDEs with Lipschitz generators. We contribute by stating and proving some comparison theorems for 2BSDEs with different generators and terminal conditions. Let us first introduce the generator of a 2BSDE and some assumptions that will ensure existence and uniqueness of solutions to those equations.

Lipschitz generators of 2BSDEs

Consider a map $H : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$. Then the convex-conjugate of H in its last argument $\gamma \in \mathbb{R}^{n\times n}$ is defined by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \Big\{ \frac{1}{2}\mathsf{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \Big\} \in \mathbb{R} \cup \{\infty\}, \quad \text{for } a \in \mathbb{S}_n^{>0}, \tag{4.3}$$

and we set $F_t(\omega, y, z, a) := +\infty$ for $a \in \mathbb{R}^{n\times n} \setminus \mathbb{S}_n^{>0}$, where $D_H \subseteq \mathbb{R}^{n\times n}$ is such that $H_t(\omega, y, z, \gamma) = +\infty$ for $\gamma \in \mathbb{R}^{n\times n} \setminus D_H$, and D_H is assumed to not-depend on (t, ω, y, z) and to contain the origin. We denote $\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t)$ and $\widehat{F}_t^0 := \widehat{F}_t(0, 0)$ for $P \in \mathcal{P}_H$. Let $D_{F_t(y,z)}$ be the domain of F in a for fixed (t, ω, y, z). To obtain existence and uniqueness of solutions to 2BSDEs with generators F we need the following combination of Assumption 2.8 and Assumption 4.1 of [STZ12]:

Assumption 4.6. (i) *$\mathcal{P}_H \neq \emptyset$ and $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z), for all $t \in [0,T]$,*

(ii) *F is $\mathbb{F}$-progressively measurable in (t, ω) for any fixed (y, z, a),*

(iii) *F is uniformly continuous in ω with respect to the supremum norm $\|\cdot\|_\infty$,*

(iv) *$\widehat{F}$ is $\mathcal{P}_H$-q.s. uniformly Lipschitz in (y, z), in the sense that $\exists\, C > 0$ s.t. $\mathcal{P}_H$-q.s. for all $y, y' \in \mathbb{R},\ z, z' \in \mathbb{R}^n$,*

$$|\widehat{F}_t(y, z) - \widehat{F}_t(y', z')| \le C\big(|y - y'| + |\widehat{a}_t^{1/2}(z - z')|\big),\ t \in [0,T].$$

(v) *$\widehat{F}^0$ satisfies $\Big(\int_0^T |\widehat{F}_s^0|^2 ds\Big)^{1/2} \in \mathbb{L}_H^2$, i.e.*

$$\sup_{P \in \mathcal{P}_H} E^P\Big[\operatorname*{ess\,sup}^P_{t\in[0,T]} E_t^{H,P}\Big(\int_0^T |\widehat{F}_s^0|^2 ds \Big)\Big] < \infty. \tag{4.4}$$

Remark 4.7. *1. Assumption 4.6, (iii) is less standard, and its importance lies in the proof of existence of solutions to 2BSDEs. In fact it provides the additional regularity of the generator needed to use the regular conditional probability distributions (shortly r.c.p.d.), which exist in the present canonical Wiener setting (see e.g. [SV79]). Using r.c.p.d. ensures a path-wise construction of solutions to 2BSDEs, i.e. without exception of negligible sets, hence avoiding any issue caused by singularity of measures in $\mathcal{P}_H$.*

2. Assumption 4.6, (v) implies in particular that

$$\sup_{P \in \mathcal{P}_H} E^P\Big[\int_0^T |\widehat{F}_s^0|^2 ds\Big] < \infty, \tag{4.5}$$

which in turn yields the integrability condition in the definition of $\mathcal{P}_H^2$ in [STZ12]. Recall that we have originally omitted (cf. Remark 4.4) this condition in the definition (4.1) of $\mathcal{P}_H$. For H such that $\widehat{F}^0$ is bounded $\mathcal{P}_H$-quasi-surely, (4.4) automatically follows.

Existence and uniqueness of solutions to 2BSDEs

Following [STZ12], a second-order BSDE is a stochastic integral equation of the type

$$Y_t = X - \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s^{\mathsf{tr}} dB_s + K_T - K_t, \quad t \in [0,T],\ \mathcal{P}_H\text{-q.s.}, \tag{4.6}$$

or equivalently

$$-dY_t = -\widehat{F}_t(Y_t, Z_t)ds - Z_t^{\mathsf{tr}} dB_t + dK_t, \quad t \in [0,T], \quad Y_T = X,\ \mathcal{P}_H\text{-q.s.}.$$

The solution to the 2BSDE (4.6) is defined as follows.

Definition 4.8. *For $X \in \mathbb{L}_H^2$, a couple $(Y,Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is called solution to the 2BSDE (4.6) if $Y_T = X$ $\mathcal{P}_H$-q.s., the process K^P defined for each $P \in \mathcal{P}_H$ by*

$$K_t^P := Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s^{tr} dB_s, \quad t \in [0,T],\ P\text{-a.s.}. \tag{4.7}$$

is P-a.s. non-decreasing, and the family $\{K^P,\ P \in \mathcal{P}_H\}$ satisfies the minimum condition

$$K_t^P = \underset{P' \in \mathcal{P}_H(t^+, P)}{\text{ess inf}^P}\, E_t^{P'}[K_T^{P'}], \quad P\text{-a.s., for all } P \in \mathcal{P}_H,\ t \in [0,T]. \tag{4.8}$$

If moreover the family $\{K^P,\ P \in \mathcal{P}_H\}$ can be aggregated into a universal process K, i.e. $K = K^P$, P-a.s. for all $P \in \mathcal{P}_H$ (see [STZ11] for more on aggregation), then one calls (Y,Z,K) solution to the 2BSDE.

The pair (F,X) will be called the parameters (generator and terminal condition) of the 2BSDE (4.6). Y will be referred to as value process and Z as the control process. The following proposition is a combination of [STZ12, Theorem 4.3, Theorem 4.6], and provides conditions for existence and uniqueness of solutions to 2BSDEs with Lipschitz generators. In addition it gives a representation of the value process of the 2BSDE in terms of the value processes (under $P \in \mathcal{P}_H$) of the associated standard BSDEs. We employ the classical notation for standard BSDEs (as in e.g. [EPQ97]) according to which the generator of the BSDE (4.10) is $-\widehat{F}$ (i.e. with a minus sign). The notation for the generator of the associated 2BSDEs however remains unchanged.

Proposition 4.9. *Let Assumption 4.6 hold. Then*

1. *Assume that $X \in \mathbb{L}_H^2$ and that $(Y,Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is a solution to the 2BSDE (4.6). Then for any $P \in \mathcal{P}_H$, Y has the representation*

$$Y_s = \underset{P' \in \mathcal{P}_H(s^+, P)}{\text{ess sup}^P}\, \mathcal{Y}_s^{P'}(t, Y_t),\ P\text{-a.s.},\ s \le t \le T, \tag{4.9}$$

where for each $P \in \mathcal{P}_H$, the couple $(\mathcal{Y}^P(\tau,\xi), \mathcal{Z}^P(\tau,\xi))$ is the unique solution to the standard BSDE with parameters $(-\widehat{F}, \xi)$:

$$\mathcal{Y}_t^P = \xi - \int_t^\tau \widehat{F}_s(\mathcal{Y}_s^P, \mathcal{Z}_s^P)ds - \int_t^\tau (\mathcal{Z}_s^P)^{\mathsf{tr}} dB_s, \quad t \leq \tau, \ P\text{-a.s.}, \tag{4.10}$$

for a $\mathbb{F}^+$-stopping time τ and $\mathcal{F}_\tau^+$-measurable random variable $\xi \in L^2(P)$. In particular, the 2BSDE (4.6) has at most one solution in $\mathbb{D}_H^2 \times \mathbb{H}_H^2$.

2. *For $X \in \mathcal{L}_H^2$, the BSDE (4.6) admits a unique solution $(Y, Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$.*

Remark 4.10. *1. From the dynamics of the value process Y, the control process Z of the 2BSDE (4.6) is uniquely given by $d\langle Y, B\rangle_t = Z_t d\langle B\rangle_t$, $\mathcal{P}_H$-q.s.. As a consequence, one can obtain Z from Y as*

$$Z_t = \limsup_{\epsilon \searrow 0} \frac{\langle Y, B\rangle_t - \langle Y, B\rangle_{t-\epsilon}}{\langle B\rangle_t - \langle B\rangle_{t-\epsilon}}, \quad t \in [0, T].$$

2. *Note that the representation (4.9) in Proposition 4.9 naturally follows from the minimum condition (4.8); this is a key step in deriving the probabilistic representation of solutions to fully nonlinear PDEs via 2BSDEs. With this at hand, uniqueness of the solution to the 2BSDE is a direct consequence of uniqueness for standard BSDEs.*

3. *As can be seen in part 2 of Proposition 4.9, sufficient conditions for existence and uniqueness of solutions to Lipschitz 2BSDEs crucially rely on the terminal X being in $\mathcal{L}_H^2$. Clearly a trivial example of random variables X that lie in $\mathcal{L}_H^2$ are those in $\mathrm{UC}_b(\Omega)$, i.e. that are uniform continuous and bounded. These include e.g. constants and also random variables that can be written as $X := g(B_{t_1}, \ldots, B_{t_k})$, with $t_1, \ldots, t_k \in [0, T]$ and some bounded uniformly continuous function $g : \mathbb{R}^{n\times k} \to \mathbb{R}$, $k \in \mathbb{N}$. This is true because since the function $\omega \to \omega(t)$ is Lipschitz continuous in the norm $\|\cdot\|_\infty$ for any $t \in [0, T]$, then X would be uniformly continuous as it is the composition of uniformly continuous functions. In particular, it is sufficient for this purpose that the function g be Lipschitz and bounded.*

Comparison theorems for 2BSDEs

As theoretical results for 2BSDEs in this chapter, we now state and prove some comparison theorems for 2BSDEs with different generators. Unlike the well-known comparison theorem for standard BSDEs, and because of the presence of the non-decreasing processes in the 2BSDE-dynamics, comparison of the generators of the 2BSDEs at one of the solutions does not suffice to imply comparison of the value processes. We distinguish two approaches: The first one leads to Proposition 4.12 and assumes that the generators of the two 2BSDEs at the solutions of one of the associated standard BSDEs are quasi-surely comparable. The second approach (cf. Theorem 4.13) rather assumes that the generators at the solution of one of the 2BSDEs are comparable and that an additional monotonicity condition on the difference of the non-decreasing processes holds, and obtain partly as a result that this difference also satisfies the minimum condition (4.8). For consistency with the current setup, our results are stated with respect to the family $\mathcal{P}_H$ of mutually singular measures, but the proofs would be analogous for the more general families of measures $\mathcal{P}_H^\kappa$ of [STZ12] as defined in (4.2). For two 2BSDEs with the same generator but different terminal conditions, a comparison theorem was stated in [STZ12, Corollary 4.4] as a by-product of the representation result (4.9). However in applications, one can sometimes be concerned with 2BSDEs with different generators. In [PZ13, Proposition 3.1], a comparison theorem is proved (as generalization of [Tev08, Theorem 2]) assuming that the difference of the non-decreasing components is also non-decreasing. Their focus is on quadratic 2BSDEs, but they also state the result for 2BSDEs with same generators. Here we prove general comparison theorems for 2BSDEs with possibly different generators. Our proofs rely on a classical linearization procedure (also used in [STZ12]) coupled with a change of measure argument. Instead of imposing specific conditions on the generators which imply existence of solutions, we only insist that we have solutions and impose conditions on the generators and other processes of interest quasi-surely. To this end, the following intermediate result will be needed.

Lemma 4.11. *Let $X \in \mathbb{L}_H^2$, λ and η be bounded $\mathbb{F}$-progressively measurable $\mathbb{R}$- and $\mathbb{R}^n$-valued processes respectively and $\varphi \in \mathbb{H}_H^2$. Let $(Y, Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ be a solution to the BSDE*

$$Y_t = X + \int_t^T (\varphi_s + \lambda_s Y_s + \eta_s^{\mathrm{tr}} \widehat{a}_s^{1/2} Z_s) ds - \int_t^T Z_s^{\mathrm{tr}} dB_s + K_T^P - K_t^P,\ t \in [0, T],\ P\text{-a.s.}, \quad (4.11)$$

with K^P nondecreasing and $K_0^P = 0$, for all $P \in \mathcal{P}_H$. If $X \geq 0$ and $\varphi_t \geq 0,\ t \in [0, T],\ P$-a.s. for all $P \in \mathcal{P}_H$ and, then $Y_t \geq 0,\ t \in [0, T],\ P$-a.s. for all $P \in \mathcal{P}_H$. If in addition $Y_0 = 0$, then $X = 0$, $\varphi_t = 0$ and $Y_t = 0,\ t \in [0, T],\ P$-a.s. for all $P \in \mathcal{P}_H$.

Proof. Let $P \in \mathcal{P}_H$ and M be defined by $M_t := \exp\Big(\int_0^t \eta_s^{\mathrm{tr}} \widehat{a}_s^{-1/2} dB_s + \int_0^t (\lambda_s + \frac{1}{2}|\eta_s|^2) ds\Big)$, P-a.s., $t \in [0, T]$. Applying Itô's product rule between t and T gives

$$M_t Y_t = M_T X - \int_t^T M_s (Z_s + Y_s \widehat{a}_s^{-\frac{1}{2}} \eta_s)^{\mathrm{tr}} dB_s + \int_t^T M_s dK_s^P + \int_t^T \varphi_s M_s ds,\ P\text{-a.s.}. \quad (4.12)$$

The process $N := \int_0^\cdot M_s (Z_s + Y_s \eta_s \widehat{a}_s^{-\frac{1}{2}})^{\mathrm{tr}} dB_s$ is a P-martingale. Indeed, under P one can write $N_t = \int_0^t M_s (\widehat{a}_s^{1/2} Z_s + Y_s \eta_s)^{\mathrm{tr}} dW_s^P,\ t \in [0, T]$, where W^P is a P-Brownian motion. Hence by Burkholder-Davis-Gundy (BDG) inequality it suffices to show that the P-expectation $E^P\Big[\big(\int_0^T M_s^2 |\widehat{a}_s^{1/2} Z_s + Y_s \eta_s|^2 ds\big)^{1/2}\Big]$ is finite. Because λ and η are bounded, $Z \in \mathbb{H}_H^2$ and

$Y \in \mathbb{D}_H^2$, it follows by BDG inequality that

$$\begin{aligned}
&E^P\Big[\big(\int_0^T M_s^2|\widehat{a}_s^{1/2}Z_s + Y_s\eta_s|^2 ds\big)^{1/2}\Big]\\
&\le E^P\Big[\sup_{s\le T} M_s\big(\int_0^T |\widehat{a}_s^{1/2}Z_s + Y_s\eta_s|^2 ds\big)^{1/2}\Big]\\
&\le E^P\big[\sup_{s\le T} M_s^2\big]^{1/2} E^P\Big[\int_0^T |\widehat{a}_s^{1/2}Z_s + Y_s\eta_s|^2 ds\Big]^{1/2}\\
&\le E^P\big[\sup_{s\le T} M_s^2\big]^{1/2}\Big(E^P\big[\int_0^T |\widehat{a}_s^{1/2}Z_s|^2 ds\big]^{1/2} + \|\eta\|_\infty T^{1/2} E^P\big[\sup_{s\le T}|Y_s|^2\big]^{1/2}\Big) < \infty,
\end{aligned}$$

where the second and third inequalities are obtained using Hölder's and Minkowski's inequalities respectively. Therefore N is a true P-martingale and taking the conditional expectation in (4.12) yields

$$Y_t = M_t^{-1}E_t^P\big[M_T X + \int_t^T M_s dK_s^P + \int_t^T \varphi_s M_s ds\big], \quad P\text{-a.s., for all } P \in \mathcal{P}_H. \tag{4.13}$$

Now if $X \ge 0$ and $\varphi \ge 0$, then it follows from (4.13) that $Y \ge 0$ (since $M > 0$ and K^P is non-decreasing). If moreover $Y_0 = 0$ then

$$E^P\big[M_T X + \int_0^T M_s dK_s^P + \int_0^T \varphi_s M_s ds\big] = 0, \quad \text{for all } P \in \mathcal{P}_H. \tag{4.14}$$

Finally since the random variable inside the expectation in (4.14) is non-negative and $M > 0$, then $X = 0$, $\varphi = 0$ and $K^P = 0$. Therefore $Y = 0$. □

Note that the proof of Lemma 4.11 does not require the minimum condition (4.8) to be satisfied for $\Big\{K^P,\ P \in \mathcal{P}_H\Big\}$. Lemma 4.11 will be used to prove the second comparison Theorem 4.13, the first being the following

Proposition 4.12. *Let X^i be in $\mathbb{L}_H^2$ and F^i be the generator associated by (4.3) to a nonlinear function H^i (for $i = 1, 2$) and satisfying (4.5) and Assumption 4.6, (i),(ii),(iv). Let $(Y^i, Z^i) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ be a solution to the 2BSDE with parameters (F^i, X^i), having the representation (4.9). Suppose*

$$X^1 \ge X^2 \quad \textit{and} \quad \widehat{F}_t^1(\mathcal{Y}_t^{2,P}, \mathcal{Z}_t^{2,P}) \le \widehat{F}_t^2(\mathcal{Y}_t^{2,P}, \mathcal{Z}_t^{2,P}),\ P\textit{-a.s., for all } t \in [0,T],\ P \in \mathcal{P}_H,$$

where $(\mathcal{Y}^{i,P}, \mathcal{Z}^{i,P})$ denotes the solution of the standard BSDE with parameters $(-\widehat{F}^i, X^i)$ under P, for $P \in \mathcal{P}_H$ (for $i = 1, 2$). Then $Y_t^1 \ge Y_t^2,\ t \in [0,T],\ P$-a.s. for all $P \in \mathcal{P}_H$.

Proof. Applying the comparison principle [EPQ97, Theorem 2.2] for standard BSDEs, one obtains $\mathcal{Y}_t^{1,P} \ge \mathcal{Y}_t^{2,P},\ P$-a.s., for all $t \in [0,T]$, for all $P \in \mathcal{P}_H$. Now for any fixed $P \in \mathcal{P}_H$, taking the essential supremum over all $P' \in \mathcal{P}_H(t^+, P)$ yields

$$Y_t^1 = \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess sup}^P} \mathcal{Y}_t^{1,P',X^1} \geq \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess sup}^P} \mathcal{Y}_t^{2,P',X^2} = Y_t^2, \quad P\text{-a.s.}, \ t \in [0,T],$$

which proves the required result by using (4.9). □

If an hypothesis on $\widehat{F}^1$ and $\widehat{F}^2$ as in Proposition 4.12 is satisfied $\mathcal{P}_H$-quasi-surely at (Y^2, Z^2), instead for all $(\mathcal{Y}^{2,P}, \mathcal{Z}^{2,P}),\ P \in \mathcal{P}_H$, then by imposing an additional monotonicity condition on the differences of the non-decreasing components of the associated 2BSDEs, we obtain the following similar result.

Theorem 4.13. *Let X^i be in $\mathbb{L}_H^2$ and F^i be the generator associated to a nonlinear function H^i $(i = 1, 2)$ and satisfying (4.5) and Assumption 4.6, (i),(ii),(iv). Let $(Y^i, Z^i) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ be a solution to the 2BSDE with parameters (F^i, X^i). Suppose*

$$X^1 \geq X^2, \quad \widehat{F}_t^1(Y_t^2, Z_t^2) \leq \widehat{F}_t^2(Y_t^2, Z_t^2), \quad \textit{for all } t \in [0,T],\ P\textit{-a.s. for all } P \in \mathcal{P}_H,$$

and $K^{1,P} - K^{2,P}$ is non-decreasing for all $P \in \mathcal{P}_H$, where $\left\{K^{i,P},\ P \in \mathcal{P}_H\right\}$ are the non-decreasing processes associated to the 2BSDEs $(F^i, X^i),\ i = 1, 2$. Then the minimum condition holds for the family $\{K^{1,P} - K^{2,P},\ P \in \mathcal{P}_H\}$, and $Y_t^1 \geq Y_t^2,\ t \in [0,T],\ P$-a.s. for all $P \in \mathcal{P}_H$.

Proof. Let $\delta Y = Y^1 - Y^2,\ \delta Z = Z^1 - Z^2$ and $\delta K = K^1 - K^2$. Then using Assumption 4.6, iv) on F^1 and the classical linearization technique, one can construct λ, η two bounded, $\mathbb{F}$-progressively measurable processes valued in $\mathbb{R}$ and $\mathbb{R}^n$ respectively such that for all $t \in [0,T]$ it holds P-a.s for any $P \in \mathcal{P}_H$ that

$$\delta Y_t = (X^1 - X^2) + \int_t^T (\delta_2 \widehat{F}_s + \lambda_s \delta Y_s + \eta_s^{\text{tr}} \widehat{a}_s^{1/2} \delta Z_s) ds - \int_t^T \delta Z_s^{\text{tr}} dB_s + \delta K_T^P - \delta K_t^P,$$

where $\delta_2 \widehat{F}_t = \widehat{F}_t^2(Y_t^2, Z_t^2) - \widehat{F}_t^1(Y_t^2, Z_t^2) \geq 0,\ t \in [0,T],\ P$-a.s. By assumption, the process $\delta K^P := K^{1,P} - K^{2,P}$ is non-decreasing and starts at 0. Moreover, $\left\{\delta K^P,\ P \in \mathcal{P}_H\right\}$ also satisfies the minimum condition (4.8). Indeed let $P \in \mathcal{P}_H$ and $t \in [0,T]$, then for all $P' \in \mathcal{P}_H(t^+, P)$ holds

$$\delta K_t^P = \delta K_t^{P'} \leq E_t^{P'}[\delta K_T^{P'}] = E_t^{P'}[K_T^{1,P'}] - E_t^{P'}[K_T^{2,P'}],\ P\text{-a.s..}$$

Taking the essential infimum over all $P' \in \mathcal{P}_H(t^+, P)$ on both sides yields

$$\begin{aligned} \delta K_t^P \leq \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess inf}^P} E_t^{P'}[\delta K_T^{P'}] &= \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess inf}^P} \left(E_t^{P'}[K_T^{1,P'}] - E_t^{P'}[K_T^{2,P'}]\right), \quad P\text{-a.s..} \\ &\leq \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess inf}^P} E_t^{P'}[K_T^{1,P'}] - \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess inf}^P} E_t^{P'}[K_T^{2,P'}], \quad P\text{-a.s.,} \end{aligned}$$

which by the minimum condition on $\{K^{1,P},\ P\in\mathcal{P}_H\}$ and $\left\{K^{2,P},\ P\in\mathcal{P}_H\right\}$ yields

$$\delta K_t^P \le \underset{P'\in\mathcal{P}_H(t^+,P)}{\text{ess inf}^P}\ E_t^{P'}[\delta K_T^{P'}] \le K_t^{1,P}-K_t^{2,P}=\delta K_t^P \quad P\text{-a.s.}.$$

This implies that $\{\delta K^P,\ P\in\mathcal{P}_H\}$ satisfies the minimum condition. By assumptions on F^1,F^2 it clearly holds $\delta_2\widehat{F}\in\mathbb{H}_H^2$. Now since $\delta_2\widehat{F}\ge 0$ and $X_1-X_2\ge 0$, then Lemma 4.11 implies $\delta Y\ge 0$. □

The following are direct consequences of Proposition 4.12 and Theorem 4.13, that could be used to describe in terms of 2BSDEs the solution to optimization problems that are stated with respect to mutually singular measures in $\mathcal{P}_H$.

Corollary 4.14. *Let $X,X^\vartheta\in\mathbb{L}_H^2$ and F,F^ϑ associated to nonlinear functions H,H^ϑ satisfying (4.5) and Assumption 4.6, (i),(ii),(iv), for ϑ in some index set Θ. Let $(Y,Z),(Y^\vartheta,Z^\vartheta)$ in $\mathbb{D}_H^2\times\mathbb{H}_H^2$ be solutions to the 2BSDEs with parameters $(F,X),(F^\vartheta,X^\vartheta)$ and having the representation (4.9). Suppose there exists $\bar\vartheta\in\Theta$ such that*

$$X=\underset{\vartheta\in\Theta}{\text{ess inf}^P}\ X^\vartheta = X^{\bar\vartheta},\quad P\text{-a.s.},\ P\in\mathcal{P}_H \quad \text{and}$$
$$\widehat{F}_t(\mathcal{Y}_t^P,\mathcal{Z}_t^P)=\underset{\vartheta\in\Theta}{\text{ess sup}^P}\ \widehat{F}_t^\vartheta(\mathcal{Y}_t^P,\mathcal{Z}_t^P)=\widehat{F}_t^{\bar\vartheta}(\mathcal{Y}_t^P,\mathcal{Z}_t^P),\quad P\text{-a.s.},\ t\in[0,T],\ P\in\mathcal{P}_H,$$

where $(\mathcal{Y}^P,\mathcal{Z}^P),(\mathcal{Y}^{\vartheta,P},\mathcal{Z}^{\vartheta,P})$ denote the solutions under P to the standard BSDEs with parameters $(-\widehat{F},X),(-\widehat{F}^\vartheta,X^\vartheta)$ respectively. Then $Y_t=\underset{\vartheta\in\Theta}{\text{ess inf}^P}\ Y_t^\vartheta=Y_t^{\bar\vartheta}$ P-a.s., for all $t\in[0,T]$, $P\in\mathcal{P}_H$.

Proof. By the hypotheses on the generators, follows $\mathcal{Y}^P=\mathcal{Y}^{\bar\vartheta,P},\ P\text{-a.s.},\ P\in\mathcal{P}_H$. This implies by the representation (4.9) that $Y=Y^{\bar\vartheta}$. Now Proposition 4.12 yields $Y_t^{\bar\vartheta}=Y_t\le Y_t^\vartheta$, P-a.s. for all $t\in[0,T]$, $P\in\mathcal{P}_H$, for all $\vartheta\in\Theta$. After taking the essential infimum over all $\vartheta\in\Theta$, this yields

$$\underset{\vartheta\in\Theta}{\text{ess inf}^P}\ Y_t^\vartheta\le Y_t^{\bar\vartheta}=Y_t\le \underset{\vartheta\in\Theta}{\text{ess inf}^P}\ Y_t^\vartheta\ P\text{-a.s.},\quad \text{for all } t\in[0,T],\ P\in\mathcal{P}_H,$$

which is the required result. □

Corollary 4.15. *Let $X,X^\vartheta\in\mathbb{L}_H^2$ and F,F^ϑ associated to nonlinear functions H,H^ϑ satisfying Assumption 4.6, for ϑ in some index set Θ. Let $(Y,Z),(Y^\vartheta,Z^\vartheta)\in\mathbb{D}_H^2\times\mathbb{H}_H^2$ be solutions to the 2BSDEs with parameters $(F,X),(F^\vartheta,X^\vartheta)$. Suppose there exists $\bar\vartheta\in\Theta$ such that*

$$X=\underset{\vartheta\in\Theta}{\text{ess inf}^P}\ X^\vartheta=X^{\bar\vartheta}\ P\text{-a.s.},\quad \text{for all } P\in\mathcal{P}_H,$$
$$\widehat{F}_t(Y_t,Z_t)=\underset{\vartheta\in\Theta}{\text{ess sup}^P}\ \widehat{F}_t^\vartheta(Y_t,Z_t)=\widehat{F}_t^{\bar\vartheta}(Y_t,Z_t),\ P\text{-a.s.},\quad \text{for all } t\in[0,T],\ P\in\mathcal{P}_H,$$

and $K^{\vartheta,P} - K^P$ is non-decreasing for all $\vartheta \in \Theta$, $P \in \mathcal{P}_H$, where $\left\{K^P,\ P \in \mathcal{P}_H\right\}$ and $\left\{K^{\vartheta,P},\ P \in \mathcal{P}_H\right\}$ are the non-decreasing components of the solutions (Y, Z) and $(Y^\vartheta, Z^\vartheta)$ for the 2BSDEs with parameters (F, X) and $(F^\vartheta, X^\vartheta)$. Then for all $t \in [0,T]$, $P \in \mathcal{P}_H$, one has

$$Y_t = \underset{\vartheta\in\Theta}{\mathit{ess\,inf}}^P\ Y_t^\vartheta = Y_t^{\bar\vartheta} \quad \text{and} \quad K_t^P = \underset{\vartheta\in\Theta}{\mathit{ess\,inf}}^P\ K_t^{\vartheta,P} = K_t^{\bar\vartheta,P},\ P\text{-a.s.}.$$

Proof. By the hypotheses on the generators it follows for all $P \in \mathcal{P}_H$ and $t \in [0,T]$ that

$$\begin{aligned} K_t^P &= Y_0 - Y_t + \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s^{\mathrm{tr}} dB_s,\ P\text{-a.s.} \\ &= Y_0 - Y_t + \int_0^t \widehat{F}_s^{\bar\vartheta}(Y_s, Z_s)ds + \int_0^t Z_s^{\mathrm{tr}} dB_s,\ P\text{-a.s.}. \end{aligned} \tag{4.15}$$

Since $X = X^{\bar\vartheta}$ and $\left\{K^P,\ P \in \mathcal{P}_H\right\}$ satisfies the minimum condition, then by uniqueness of solutions to 2BSDEs, (4.15) implies $(Y, Z) = (Y^{\bar\vartheta}, Z^{\bar\vartheta})$. This yields $K_t^P = K_t^{\bar\vartheta,P}$, P-a.s., $t \in [0,T]$, for all $P \in \mathcal{P}$. Moreover by Theorem 4.13 one obtains from the hypotheses on the generators and the associated non-decreasing processes that $Y_t \le Y_t^\vartheta$ holds P-a.s., for all $t \in [0,T]$, $P \in \mathcal{P}_H$, for all $\vartheta \in \Theta$. Now taking the essential infimum over all $\vartheta \in \Theta$ yields

$$\underset{\vartheta\in\Theta}{\text{ess inf}}^P\ Y_t^\vartheta \le Y_t^{\bar\vartheta} = Y_t \le \underset{\vartheta\in\Theta}{\text{ess inf}}^P\ Y_t^\vartheta \quad P\text{-a.s., for all } t \in [0,T],\ P \in \mathcal{P}_H.$$

Furthermore, observe that since for all $\vartheta \in \Theta$ the process $K^{\vartheta,P} - K^P$ is non-decreasing P-a.s., then the process $\underset{\vartheta\in\Theta}{\text{ess inf}}^P\ K^{\vartheta,P} - K^P$ is also P-a.s. non-decreasing and starts at 0 for all $P \in \mathcal{P}_H$. In addition, since $K^P = K^{\bar\vartheta,P}$, for all $P \in \mathcal{P}_H$, then the inequalities

$$0 \le \underset{\vartheta\in\Theta}{\text{ess inf}}^P\ K_t^{\vartheta,P} - K_t^P \le \underset{\vartheta\in\Theta}{\text{ess inf}}^P\ K_t^{\vartheta,P} - K^{\bar\vartheta,P} \le 0.$$

hold. ☐

4.2 Market model and good-deal constraint under volatility uncertainty

We apply the preceding 2BSDE theory to good-deal valuation and hedging of contingent claims in incomplete financial markets under volatility uncertainty. Recall (cf. [DM06, DK13a, NS12, EJ13, EJ14, Vor14]) that in the framework of volatility uncertainty, the reference probability measures interpreted as generalized scenarios in the market (cf. [ADE+07]) are no longer dominated and may actually be mutually singular. In comparison to standard BSDEs which are used in Chapters 3 and 2 in the presence of drift uncertainty or absence of uncertainty at all, 2BSDEs seem to be an appropriate tool for describing worst-case valuations in the presence of volatility uncertainty (see also [MPZ15]). We will characterize worst-case good-deal bounds and associated robust hedging strategies via solutions to 2BSDEs. As in [CR00, BS06], we consider good-deal constraints imposed as bounds on the Sharpe ratios (equivalently bounds on the optimal growth rates as in [Bec09]) in the financial market extended by additional wealth processes. First let us specify the model for the market with uncertainty about the volatility.

4.2.1 Financial market with volatility uncertainty

The financial market consists of d tradeable stocks ($d \leq n$) with discounted price processes $(S^i)_{i=1}^d = S$ modelled by

$$dS_t = \operatorname{diag}(S_t)\sigma_t dB_t,\ t \in [0,T],\ \mathcal{P}_H\text{-q.s.},\quad S_0 \in (0,\infty)^d, \tag{4.16}$$

where σ is a $\mathbb{R}^{d\times n}$-valued $\mathbb{F}$-predictable process, each σ_t being uniformly continuous in ω with respect to $\|\cdot\|_\infty$. We assume that $\sigma\sigma^{\mathrm{tr}}$ is uniformly bounded and uniformly elliptic, i.e.

$$\text{there exists } K, L > 0 \text{ such that}\quad K\,\mathsf{I}_d \leq \sigma\sigma^{\mathrm{tr}} \leq L\,\mathsf{I}_d,\quad \mathcal{P}_H \otimes dt\text{-q.s.}, \tag{4.17}$$

where I_d denotes the $d\times d$ identity matrix. In particular $\sigma\widehat{a}^{1/2}$ is $\mathcal{P}_H \otimes dt$-q.s. of maximal rank $d \leq n$, since $\sigma_t\widehat{a}_t\sigma_t^{\mathrm{tr}}$ is uniformly elliptic and bounded (using (4.1) and (4.17)).

Remark 4.16. *1. From (4.17) and (4.1) holds* $\sup_{P\in\mathcal{P}_H} E^P\big[\int_0^T |\sigma_t\bar{a}^{1/2}|^2 dt\big] < \infty$*, and hence by [DM06, Lemma 2.4 and Theorem 2.8] the family* $\left\{{}^{(P)}\!\int \sigma_s dB_s,\ P \in \mathcal{P}_H\right\}$ *of stochastic integrals can be aggregated into a single process* $\int_0^\cdot \sigma_s dB_s$ *that is defined* $\mathcal{P}_H$*-quasi-surely. In fact under additional assumptions (e.g. càdlàg integrands as in [Kar95], or continuum hypothesis as in [Nut12b]) they can even be defined path-wise without exception of a null-set.*

2. *The market model captures uncertainty about the volatility in the sense that under each measure* $P \in \mathcal{P}_H$*, one has* $dB_t = \widehat{a}_t^{1/2} dW_t^P$*, where* W^P *is a* P*-Brownian motion. In fact, substituting this in the dynamics of* S *in (4.16) one sees that under the reference measure* $P \in \mathcal{P}_H$*, the process* $\sigma\widehat{a}^{1/2}$ *plays the role of the instantaneous volatility matrix for the stock prices* S*. In this sense, Knightian uncertainty (ambiguity) about future volatility scenarios is captured by the local martingale laws* $P \in \mathcal{P}_H$ *for* S*.*

3. *The bounds* $\underline{a}, \bar{a}$ *and the uniform bounds on* $\sigma\sigma^{tr}$ *can be viewed as setting a confidence region for future volatility values, calibrated e.g. from extreme implied (or historical) volatilities in the market.*

4. *The financial market described is incomplete under any scenario $P \in \mathcal{P}_H$ for the volatility $\sigma\widehat{a}^{1/2}$ if $d < n$, since $\sigma\widehat{a}^{1/2}$ is of full rank $\mathcal{P}_H \otimes dt$-q.s..*

Let $\mathcal{M}^e(P)$ be the set of equivalent local martingale measures of S under each model P, for $P \in \mathcal{P}_H$. Denoting ${}^{(P)}\mathcal{E}(M) := \exp\left(M - M_0 - \frac{1}{2}\langle M\rangle^P\right)$ the stochastic exponential of the local P-martingale M under P, we have the following

Lemma 4.17. *For $P \in \mathcal{P}_H$, the set $\mathcal{M}^e(P)$ consists of the equivalent measures $Q \sim P$ such that $dQ = {}^{(P)}\mathcal{E}(\eta \cdot W^P)dP$ with η $\mathbb{F}$-progressively measurable and $\eta_t \in \operatorname{Ker}(\sigma_t\widehat{a}_t^{1/2})$, $t \in [0,T]$.*

Proof. Let $P \in \mathcal{P}_H$. By the martingale representation theorem under P (see Lemma 4.2), any $Q \in \mathcal{M}^e(P)$ satisfies $dQ = {}^{(P)}\mathcal{E}(\eta \cdot W^P)dP$ for a $\mathbb{F}$-progressive measurable process η such that $\int_0^T |\eta_s|^2 ds < \infty,\ P$-a.s. holds. By Girsanov theorem, one can rewrite the dynamics of S under Q as $dS_t = \operatorname{diag}(S_t)\left(\sigma_t\widehat{a}_t^{1/2}\eta_t dt + \sigma_t\widehat{a}_t^{1/2}dW_t^Q\right)$, $t \in [0,T]$, where $W^Q = W^P - \int_0^\cdot \eta_s ds$ is a Q-Brownian motion by Levy's characterization. Now Q is a local martingale measure for S if and only if $\sigma_t\widehat{a}_t^{1/2}\eta_t = 0$ for all $t \in [0,T]$, i.e. if and only if $\eta_t \in \operatorname{Ker}(\sigma_t\widehat{a}_t^{1/2}),\ t \in [0,T]$. □

Remark 4.18. 1. *Note from (4.16) that the measures $P \in \mathcal{P}_H$ are also local martingale measures for S. This implies that $P \in \mathcal{M}^e(P) \neq \emptyset$, for any $P \in \mathcal{P}_H$. As a consequence, the market satisfies the no-free lunch with vanishing risk condition (see [DS94]) under each $P \in \mathcal{P}_H$. This is equivalent to a robust notion for no-arbitrage under uncertainty (see [BBKN14]).*

2. *Modeling the stock price process directly under local-martingale measures (i.e. setting its drift to zero) is a technical assumption rather than financially justified. In our case, this will ensure convexity (in $a \in \mathbb{S}_n^{>0}$) of the generators F of the upcoming pricing and hedging 2BSDEs. This convexity is essential in 2BSDE theory since F is defined by (4.3) as the convex conjugate of a function H. Confer part 1 of Remark 4.21 for further notes about the possible limitations for the applicability of 2BSDE theory if one includes a non-zero drift in (4.16) .*

We parametrize trading strategies $\varphi = (\varphi^i)_{i=1}^d$ in terms of the amount φ^i of wealth invested in the stock with price process S^i, with φ being a $\mathbb{F}^+$-progressively measurable process with suitable integrability properties. In this respect, the wealth process V^φ associated to a trading strategy φ with initial capital V_0 (so that (V_0, φ) quasi-surely satisfies the self-financing requirement) has the dynamics

$$V_t^\varphi = V_0 + \int_0^t \varphi_s^{\mathrm{tr}}\sigma_s dB_s, \quad t \in [0,T],\ \mathcal{P}_H\text{-q.s..}$$

Re-parameterizing trading strategies in terms of integrands $\phi := \sigma^{\mathrm{tr}}\varphi \in \operatorname{Im}\sigma^{\mathrm{tr}}$ with respect to B, the dynamics of the wealth process $V^\phi := V^\varphi$ rewrites

$$V_t^\phi = V_0 + \int_0^t \phi_s^{\mathrm{tr}} dB_s = V_0 + {}^{(P)}\!\!\int_0^t \phi_s^{\mathrm{tr}} \widehat{a}_s^{\frac{1}{2}} dW_s^P, \; P\text{-a.s.}, \; t \in [0,T], \; P \in \mathcal{P}_H. \tag{4.18}$$

We denote $\Phi(P), \; P \in \mathcal{P}_H$ the set of trading strategies that are permitted under P (referred to as P-permitted), defined as

$$\Phi(P) := \Big\{\phi : \phi \text{ is } \mathbb{F}^+\text{-prog. meas.}, \; E^P\big[\int_0^T |\widehat{a}_s^{1/2}\phi_s^{\mathrm{tr}}|^2 ds\big] < \infty, \text{ and } \phi \in \operatorname{Im}\sigma^{\mathrm{tr}}\Big\},$$

with "prog. meas." abbreviating progressively measurable. We use the following definition of the set of permitted trading strategies.

Definition 4.19. *The set Φ of permitted trading strategies under volatility uncertainty consists of all $\mathbb{F}^+$-progressively measurable processes $\phi \in \operatorname{Im}\sigma^{tr}$ satisfying*

$$\sup_{P \in \mathcal{P}_H} E^P\big[\int_0^T |\widehat{a}_s^{1/2}\phi_s^{tr}|^2 ds\big] < \infty$$

and such that the family of stochastic integrals $\Big\{{}^{(P)}\!\!\int_0^\cdot \phi_s^{tr} dB_s, \, P \in \mathcal{P}_H\Big\}$ aggregates into a single process $\int_0^\cdot \phi_s^{tr} dB_s$.

By its quasi-sure definition, the integral $\int_0^\cdot \phi_s^{\mathrm{tr}} dB_s$, for a strategy $\phi \in \Phi$, satisfies $\int_0^\cdot \phi_s^{\mathrm{tr}} dB_s = {}^{(P)}\!\!\int_0^\cdot \phi_s^{\mathrm{tr}} dB_s, \; P$-a.s. for all $P \in \mathcal{P}_H$. The trading strategies in Φ are termed as $\mathcal{P}_H$-permitted (or simply permitted). Clearly V^ϕ is a P-martingale for any $\phi \in \Phi \subseteq \bigcap_{P \in \mathcal{P}_H} \Phi(P)$ and $P \in \mathcal{P}_H$, hence excluding existence of arbitrage strategies in Φ for any scenario $\sigma\widehat{a}^{1/2}$ of the volatility.

4.2.2 No-good-deal constraint

In the absence of uncertainty, we consider a no-good-deal constraint defined as a bound on the instantaneous Sharpe ratios, for any market extension by additional derivative price processes obtained from the no-good-deal pricing measures (cf. [CR00, BS06] and references therein). This no-good-deal constraint is equivalent to a bound on the optimal expected growth rates of returns, again in any market extension (see [Bec09]). Classically, such can be ensured (using the Hansen-Jagannathan inequality) by imposing a bound on the norm of Girsanov kernels for risk-neutral pricing measures. In the presence of drift (rather than volatility) uncertainty, results about good-deal valuation and robust hedging are provided in Chapter 3. Our aim here is to derive analogs of these results in the presence of volatility uncertainty. The no-good-deal constraint under volatility uncertainty consists of imposing the same bound h on the Girsanov kernels of pricing measures in every model $P \in \mathcal{P}_H$ separately. By doing this, we obtain for each P a set of no-good-deal measures $\mathcal{Q}^{\text{ngd}}(P) \subseteq \mathcal{M}^e(P)$. Following a worst-case approach to good-deal valuation under uncertainty (as in (3.49) in Chapter 3, but taking into account here the possible singularity of the priors $P \in \mathcal{P}_H$), this will yield a larger good-deal bound obtained as the supremum of prices taken over all no-good-deal measures for all reference measures $P \in \mathcal{P}_H$. To be more precise let h be a fixed positive bounded $\mathbb{F}$-progressively measurable process that is uniformly continuous in ω with respect to $\|\cdot\|_\infty$. We consider the set $\mathcal{Q}^{\text{ngd}}(P)$ of no-good-deal measures in the model $P \in \mathcal{P}_H$ as the subset of $\mathcal{M}^e(P)$ consisting of equivalent local martingale measures Q, whose Girsanov kernels η with respect to the P-Brownian motion W^P are bounded by h, i.e. $|\eta_t(\omega)| \leq h_t(\omega)$ for all $(t,\omega) \in [0,T] \times \Omega$. In other words, using Lemma 4.17, we define

$$\mathcal{Q}^{\text{ngd}}(P) := \Big\{ Q \sim P \;\Big|\; dQ/dP = {}^{(P)}\mathcal{E}\left(\eta \cdot W^P\right), \text{ with } \mathbb{F}\text{-prog. meas. } \eta \\ \text{satisfying } \eta \in \operatorname{Ker}(\sigma \widehat{a}^{1/2}) \text{ and } |\eta| \leq h \Big\}.$$

Clearly, for all $P \in \mathcal{P}_H$ holds $P \in \mathcal{Q}^{\text{ngd}}(P) \neq \emptyset$. Note that uniform continuity of h and σ will ensure that the forthcoming 2BSDE generators satisfy Assumption 4.6, iii), needed for wellposedness of the associated 2BSDEs (see Theorem 4.9). As in part b) of Lemma 3.1 in Chapter 3, one can show that for $P \in \mathcal{P}_H$ the set $\mathcal{Q}^{\text{ngd}}(P)$ is convex and multiplicatively stable (in short *m-stable*). M-stability of a set of priors is usually key for obtaining time-consistency of the corresponding process dynamically defined as essential supremum over conditional expectations over the priors; see [Del06] for the definition and a general study of m-stability when the priors are dominated. M-stability is also referred to as rectangularity in the economic literature [CE02].

4.3 Good-deal bounds and hedging under volatility uncertainty

Using 2BSDEs, we describe good-deal bounds in the market model of Section 4.2.1 and study an associated notion of robust hedging in the framework of volatility uncertainty. We first define the good-deal valuation bounds whose financial motivation comes from the no-good-deal restriction mentioned previously. Then we characterize the corresponding good-deal bounds in terms of solutions to Lipschitz 2BSDEs. After that, we derive hedging strategies as minimizers of some dynamic coherent a-priori risk measure ρ under volatility uncertainty (e.g. as in [NS12]), so that the good-deal bound arises as the market consistent risk measure associated to ρ, in the spirit of [BE09]. Our definition of the good-deal bounds and hedging strategies will take into account the dependence of the no-good-deal restriction on the prior, and the aversion of investors to volatility uncertainty.

4.3.1 Good-deal bounds under volatility uncertainty

As in Chapter 3, Section 3.3, the main idea behind good-deal valuation under uncertainty is to view aversion to model uncertainty as a penalization of the no-good-deal restriction yielding a larger good-deal bound than in the absence of uncertainty. We use a worst-case approach to uncertainty aversion in the spirit of [GS89, HS01, CE02]. This approach has been used for example in [ALP95, Lyo95, NS12, Vor14] to study robust arbitrage bounds and super-hedging strategies in a financial market with volatility uncertainty or in [SW05, Sch07, Que04, DK13a, MPZ15] for robust utility maximization under model uncertainty. Intuitively, an uncertainty-averse investor faced with insufficient knowledge about the actual financial market volatility, would opt for a worst-case approach to valuation in order to compensate for eventual losses due to the wrong choice of the volatility. Acting this way, she would sell (resp. buy) financial risks at the largest (resp. smallest) good-deal bounds over all possible scenarios in her confidence set of volatility values, corresponding to the set $\mathcal{P}_H$ of reference priors. Acknowledging that mutual singularity of the reference measures in $\mathcal{P}_H$ brings additional technical difficulties in making rigorous sense of essential suprema, we define the (robust) worst-case good-deal bound $\pi^u_\cdot(X)$ in our dynamic framework for a financial risk $X \in L^2_H$ as the unique process $\pi^u_\cdot(X) \in \mathbb{D}^2_H$ (if it exists) that satisfies

$$\pi^u_t(X) = \underset{P'\in\mathcal{P}_H(t^+,P)}{\text{ess sup}^P}\ \underset{Q\in\mathcal{Q}^{\text{ngd}}(P')}{\text{ess sup}^{P'}} E^Q_t[X],\ t\in[0,T],\ P\text{-a.s.},\quad \text{for all } P\in\mathcal{P}_H. \tag{4.19}$$

The definition of the lower good-deal bound $\pi^l_\cdot(X) = -\pi^u_\cdot(-X)$ is analogous, replacing the essential suprema in (4.19) by essential infima; for this reason we focus only on studying the upper bound. For $X \in \mathcal{L}^2_H$, the good-deal bound $\pi^u_\cdot(X)$ will be shown to be a single universal process corresponding to the Y-component of the solution of a 2BSDE.

Before proceeding, let us introduce some notations that will be used throughout the sequel. For $a \in \mathbb{S}^{>0}_n$, we denote by $\Pi^a_t(\cdot)$ and $\Pi^{a,\perp}_t(\cdot)$ respectively the orthogonal projections onto the subspaces $\mathrm{Im}\,(\sigma_t a^{1/2})^{\mathsf{tr}}$ and $\mathrm{Ker}\,(\sigma_t a^{1/2})$ of $\mathbb{R}^n$, $t\in[0,T]$. More precisely for each $a \in \mathbb{S}^{>0}_n$ and $t\in[0,T]$, we define the projections of $z\in\mathbb{R}^n$ as

$$\Pi^a_t(z) = (\sigma_t a^{1/2})^{\mathsf{tr}}(\sigma_t a \sigma_t^{\mathsf{tr}})^{-1}(\sigma_t a^{1/2})z \quad \text{and} \quad \Pi^{a,\perp}_t(z) = z - \Pi^a_t(z). \tag{4.20}$$

In particular we define (in a path-wise sense) $\widehat{\Pi}_t(\cdot) := \Pi^{\widehat{a}_t}_t(\cdot)$ and $\widehat{\Pi}^\perp_t(\cdot) := \Pi^{\widehat{a}_t,\perp}_t(\cdot)$. For each $P\in\mathcal{P}_H,\ t\in[0,T]$ and $P'\in\mathcal{P}_H(t^+,P)$, the standard good-deal bound in the model P' is given as usual by $\pi^{u,P'}_t(X) := \text{ess sup}^{P'}_{Q\in\mathcal{Q}^{\text{ngd}}(P')} E^Q_t[X],\ P$-a.s., so that by (4.19) one has

$$\pi^u_t(X) = \underset{P'\in\mathcal{P}_H(t^+,P)}{\text{ess sup}^P}\ \pi^{u,P'}_t(X),\ P\text{-a.s.},\ t\in[0,T],\ P\in\mathcal{P}_H,\ \text{for } X\in\mathbb{L}^2_H. \tag{4.21}$$

Note from Theorem 3.15 in Chapter 3 that the good-deal bound $\pi^{u,P'}_\cdot(X)$ for $P'\in\mathcal{P}_H(t^+,P)$ and $P\in\mathcal{P}_H$ is the value process of the standard BSDE under P with generator $-\widehat{F}_t(\cdot) =$

$-F_t(\cdot, \widehat{a}_t),\ t \in [0,T]$, and terminal condition X, with F given for $z \in \mathbb{R}^n, a \in \mathbb{R}^{n\times n}$ by

$$F(t,z,a) = \begin{cases} -h_t|\Pi_t^{a,\perp}(a^{\frac{1}{2}}z)| & \text{if } a \in \mathbb{S}_n^{>0} \text{ and } \underline{a} \leq a \leq \overline{a}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.22}$$

For $X \in \mathbb{L}_H^2$, we consider the 2BSDE

$$Y_t = X - \int_t^T Z_s^{\mathsf{tr}} dB_s - \int_t^T \widehat{F}_s(Z_s)ds + K_T - K_t, \quad t \in [0,T], \quad \mathcal{P}_H\text{-q.s.}, \tag{4.23}$$

where $\widehat{F}_t(\cdot) := F_t(\cdot, \widehat{a}_t)$ for F given by (4.22). Using (4.19) and the representation formula in Proposition 4.9, we show the following

Theorem 4.20. *1. If $X \in \mathbb{L}_H^2$ and $(Y,Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is a solution to the 2BSDE (4.23), then the good-deal bound is uniquely given by $\pi_t^u(X) = Y_t,\ t \in [0,T],\ \mathcal{P}_H$-q.s. and satisfies*

$$\pi_t^u(X) = Y_t = \operatorname*{ess\,sup^{\it P}}_{P' \in \mathcal{P}_H(t^+,P)} \operatorname*{ess\,sup^{\it P'}}_{Q \in \mathcal{Q}^{ngd}(P')} E_t^Q[X], \quad t \in [0,T],\ P\text{-a.s. for all } P \in \mathcal{P}_H.$$

2. For $X \in \mathcal{L}_H^2$, there exists a unique solution $(Y,Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ to the 2BSDE (4.23).

Proof. For $z \in \mathbb{R}^n,\ t \in [0,T]$, the generator $F(t,z,a)$ writes explicitly for $a \in \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$ as

$$F(t,z,a) = -h_t\Big(z^{\mathsf{tr}}(a - a\sigma_t^{\mathsf{tr}}(\sigma_t a \sigma_t^{\mathsf{tr}})^{-1}\sigma_t a))z\Big)^{1/2}.$$

First we need to show that the function $F(t,z,\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}$ is convex on its domain $D_{F_t} = \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$, from which the Fenchel-Moreau theorem would imply that $F(t,z,\cdot)$ is the convex conjugate of a nonlinear function H such that (4.3) holds. For this purpose, it suffices to show that the function $G_t : a \mapsto a\sigma_t^{\mathsf{tr}}(\sigma_t a\sigma_t^{\mathsf{tr}})^{-1}\sigma_t a$ is $\mathbb{S}_n^{>0}$-convex. Let then $\mu \in [0,1]$ and $a, \tilde{a} \in \mathbb{S}_n^{>0}$. Using the Schur complement condition for positive semi-definiteness [HJ12, Theorem 7.7.7 or Theorem 7.7.16], convexity of G_t is equivalent to positive semi-definiteness of the matrix $A_t \in \mathbb{R}^{(n+d)\times(n+d)}$ given by

$$\begin{aligned} A_t &= \begin{pmatrix} \mu\, a\sigma_t^{\mathsf{tr}}(\sigma_t a\sigma_t^{\mathsf{tr}})^{-1}\sigma_t a + (1-\mu)\,\tilde{a}\sigma_t^{\mathsf{tr}}(\sigma_t\tilde{a}\sigma_t^{\mathsf{tr}})^{-1}\sigma_t\tilde{a} & \big(\sigma_t(\mu a + (1-\mu)\tilde{a})\big)^{\mathsf{tr}} \\ \sigma_t(\mu a + (1-\mu)\tilde{a}) & \sigma_t(\mu a + (1-\mu)\tilde{a})\sigma_t^{\mathsf{tr}} \end{pmatrix} \\ &= \mu \begin{pmatrix} a\sigma_t^{\mathsf{tr}}(\sigma_t a\sigma_t^{\mathsf{tr}})^{-1}\sigma_t a & (\sigma_t a)^{\mathsf{tr}} \\ \sigma_t a & \sigma_t a\sigma_t^{\mathsf{tr}} \end{pmatrix} + (1-\mu) \begin{pmatrix} \tilde{a}\sigma_t^{\mathsf{tr}}(\sigma_t\tilde{a}\sigma_t^{\mathsf{tr}})^{-1}\sigma_t\tilde{a} & (\sigma_t\tilde{a})^{\mathsf{tr}} \\ \sigma_t\tilde{a} & \sigma_t\tilde{a}\sigma_t^{\mathsf{tr}} \end{pmatrix} \\ &=: \mu A_t^1 + (1-\mu)A_t^2. \end{aligned}$$

Now since $\sigma_t a \sigma_t^{\mathsf{tr}}$ and $\sigma_t \tilde{a} \sigma_t^{\mathsf{tr}}$ are positive definite and the set of positive semi-definite matrices is a convex cone, then the Schur complement condition applied to A_t^1 and A_t^2 implies that A_t is positive semi-definite.

For existence and uniqueness of the solution to the 2BSDE (4.23) we aim to apply part 2 of Proposition 4.9. To this end, we show that F satisfies parts (i)-(v) of Assumption 4.6. Part (i) is clear by definition of F in (4.22) and the fact that $D_{F_t} = \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$. As for part (ii), it holds from the progressive measurability of the processes σ and h. To show that part (iii) about uniform continuity of F holds, recall that the point-wise product of two bounded uniformly continuous functions is uniformly continuous, and that the composition of two uniformly continuous functions is also uniformly continuous. With this it follows that F is uniformly continuous in ω with respect to $\|\cdot\|_\infty$ for fixed $(t,z,a) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, since h and σ are uniformly continuous and bounded, $\sigma\sigma^{\mathsf{tr}}$ and $\sigma a \sigma^{\mathsf{tr}}$ are uniformly elliptic and bounded in the matrix norm, and the square root function is uniformly continuous. Since $\widehat{F}^0 = 0$, then part (v) obviously holds. It remains to show part (iv) about the Lipschitz continuity of $\widehat{F}$ in z. By the Minkowski inequality one has $\mathcal{P}_H$-q.s. for all $t \in [0,T]$, that

$$|\widehat{F}_t(z) - \widehat{F}_t(z')| = h_t \Big| |\widehat{\Pi}_t^{\perp}(\widehat{a}_t^{\frac{1}{2}} z)| - |\widehat{\Pi}_t^{\perp}(\widehat{a}_t^{\frac{1}{2}} z')| \Big| \leq h_t |\widehat{\Pi}_t^{\perp}(\widehat{a}_t^{\frac{1}{2}}(z-z'))| \leq \|h\|_\infty |\widehat{a}_t^{\frac{1}{2}}(z-z')|$$

holds. Hence part (iv) follows and this concludes that F satisfies Assumption 4.6.

Part 1: Recall from the discussion preceding the statement of the theorem that $\pi_\cdot^{u,P'}(X)$ for $P' \in \mathcal{P}_H(t^+, P)$ and $P \in \mathcal{P}_H$ solves the standard BSDE with generator $\widehat{F}_t(\cdot) = F_t(\cdot, \widehat{a}_t)$, $t \in [0,T]$, under P, for F given by (4.22). Part 1 is now a direct consequence of part 1 of Proposition 4.9, and the definition (4.19) of the good-deal bound $\pi_\cdot^u(X)$.

Part 2: Direct application of part 2 of Proposition 4.9 gives the claim. □

Remark 4.21. *1. Were the dynamics (4.16) of the stock price processes rather given by the SDE $dS_t = \mathrm{diag}(S_t)(b_t dt + \sigma_t dB_t)$, with non-zero drift b, a candidate for the generator of the 2BSDE (4.23) would have been by Theorem 3.15 in Chapter 3 given for $t \in [0,T], z \in \mathbb{R}^n, a \in \mathbb{R}^{n\times n}$ as*

$$F(t,z,a) = \begin{cases} \xi_t^{a\,tr} \Pi_t^a(a^{\frac{1}{2}} z) - \left(h_t^2 - |\xi_t^a|^2\right)^{1/2} |\Pi_t^{a,\perp}(a^{\frac{1}{2}} z)| & \text{if } a \in \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}] \\ +\infty & \text{otherwise,} \end{cases} \tag{4.24}$$

with $\xi_t^a := (\sigma_t a^{1/2})^{tr} (\sigma_t a \sigma_t^{tr})^{-1} b_t \in \mathrm{Im}\,(\sigma_t a^{1/2})^{tr}$ being the market price of risk in a model with volatility $\sigma a^{1/2}$. Clearly this involves an additional dependence of F in $a \in \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$, for which it becomes very difficult to see whether F is convex or not in $a \in \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$. Indeed a sufficient condition for the convexity of $F(t,z,\cdot)$ given by (4.24) is that each summand is convex in $a \in \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$. However, the second summand of F is a product of two functions, and would be convex if the two components of the product are convex and either monotone increasing or monotone decreasing functions in

$a \in \mathbb{S}_n^{>0}$ (cf. [BV04, Exercise 3.32]). But we have not been able to verify these properties and also the Schur complement condition is no longer enough to show convexity of the product. For these technical reasons we have modelled S directly as local martingale measures under $P \in \mathcal{P}_H$, i.e. with zero drift $b = 0$ (cf. (4.16) and Remark 4.18).

2. *Theorem 4.20 shows in particular that the family of essential supremums in (4.19) indexed by the measures $P \in \mathcal{P}_H$ effectively aggregates into a single process $\pi^u_\cdot(X)$. In fact using r.c.p.d. $\pi^u_\cdot(X)$ can be constructed without exception of a null-set, for $X \in \mathrm{UC_b}(\Omega)$ and then extended by density to $X \in \mathcal{L}^2_H$ (see [STZ13]). Moreover [STZ13, Proposition 4.11] implies that $\pi^u_\cdot(X)$ is actually $\mathbb{F}$-progressively measurable, for $X \in \mathcal{L}^2_H$. Hence by the Blumenthal Zero-One law (cf. Lemma 4.2) $\pi^u_0(X)$ is constant and given by $\pi^u_0(X) = \sup_{P\in\mathcal{P}_H} \pi^{u,P}_0(X)$.*

3. *By [STZ13, Proposition 4.7], the good-deal bound $\pi^u_\cdot(\cdot)$ satisfies a dynamic programing principle (recursiveness): for all $s \le t \le T$, $X \in \mathcal{L}^2_H$, holds P-a.s. for all $P \in \mathcal{P}_H$ that*

$$\pi^u_s(X) = \underset{P'\in\mathcal{P}_H(s+,P)}{\text{ess sup}^P}\, \pi^{u,P'}_s(\pi^u_t(X)) = \underset{P'\in\mathcal{P}_H(s+,P)}{\text{ess sup}^P}\ \underset{Q\in\mathcal{Q}^{ngd}(P')}{\text{ess sup}^{P'}}\, E^Q_s[\pi^u_t(X)] = \pi^u_s(\pi^u_t(X)).$$

   This is equivalent to a time consistency property of the process $\pi^u_\cdot(X)$, for $X \in \mathcal{L}^2_H$.

4. *Using Proposition 4.12, it holds analogously to Lemma 3.1 in Chapter 3 (see also [KS07b, Theorem 2.7] or [Bec09, Proposition 2.6]) that the good-deal bound $\pi^u_\cdot(X)$ satisfies the properties of dynamic coherent risk measures (with generalized scenarios consisting of measures that can be associated to volatility uncertainty). In addition by part 2. it is time-consistent. These facts will be used to define good-deal hedging in terms of minimization of a risk measure of the type of $\pi^u_\cdot(\cdot)$. We refer to [NS12] for a general study of dynamic risk measures under volatility uncertainty. Note that our subsequent results on hedging are, differently from [NS12], not on superhedging.*

Remark 4.22. *We are not able to give more general examples of elements in $\mathcal{L}^2_H$ than those provided in part 3 of Remark 4.10. This is restrictive for financial applications where one would typically be interested in X being contingent claims that have some exponential dependence in B_T and $\langle B\rangle_T$, e.g. $X = \left(\mathcal{K} - \exp\left(B_T - \langle B\rangle_T/2\right)\right)^+ \in \mathbb{L}^2_H$ in dimension $n = 1$ modeling a put option with strike $\mathcal{K} > 0$ on a Black-Scholes risky asset with uncertain volatility. Clearly, this Markovian claim does not fit into the examples given in part 3 of Remark 4.10. Fortunately for some 2BSDE generators one can sometimes identify the solution to the 2BSDE via PDE arguments, even if $X \in \mathbb{L}^2_H$ does not belong to $\mathcal{L}^2_H$; cf. e.g. Section 4.3.3.*

4.3.2 Robust good-deal hedging under volatility uncertainty

Our aim now is to define and characterize the good-deal hedging strategy using solutions to 2BSDEs. Here the objective of the investor is to find a $\mathcal{P}_H$-permitted trading strategy that minimizes her residual risk (measured under some risk measure ρ) from any time onward when holding a liability X and trading dynamically in the market. Since the investor (say the seller) requires the premium $\pi^u_\cdot(X)$ for X, then she would like the good-deal valuation to be the minimal capital requirement to make her position acceptable. In this sense, the good-deal bound would be the market consistent risk measure associated to good-deal hedging via ρ; cf. [BE09]. The risk to be minimized is measured in terms of a dynamic risk measure compatible with the no-good-deal constraint in the market and the uncertainty-aversion of the investor. The second objective of the investor should be towards robustness (of hedges and valuations) with respect to volatility uncertainty. As in Proposition 3.25 of Chapter 3 we show robustness of the good-deal hedging strategy as a supermartingale property of its tracking (hedging) error with respect to a class of a-priori valuation measures $\mathcal{P}^{\mathsf{ngd}} \supseteq \cup_{P\in\mathcal{P}_H}\mathcal{Q}^{\mathsf{ngd}}(P)$, i.e. uniformly over all reference models $P \in \mathcal{P}_H$. Recalling the definition of $\pi^u_\cdot(X)$ (for $X \in \mathbb{L}^2_H$) in (4.19) and previous results on good-deal valuation and hedging in the absence of model uncertainty (cf. e.g. [Bec09, Theorem 5.4] or Theorem 3.17 in Chapter 3), one has for all $P \in \mathcal{P}_H$, and $P' \in \mathcal{P}_H(t^+, P)$

$$\pi^{u,P'}_t(X) = \underset{\phi\in\Phi(P')}{\operatorname{ess\,inf}}^P \rho^{P'}_t\Big(X - \int_t^T \phi^{\mathsf{tr}}_s dB_s\Big), \quad P\text{-a.s.}, \ t \in [0,T], \tag{4.25}$$

where for $P \in \mathcal{P}_H$ we define $\rho^P_t(X) := \underset{Q\in\mathcal{P}^{\mathsf{ngd}}(P)}{\operatorname{ess\,sup}}^P E^Q_t[X], \ P\text{-a.s.}, \ t \in [0,T]$ with

$$\mathcal{P}^{\mathsf{ngd}}(P) := \Big\{Q \sim P \mid dQ/dP = {}^{(P)}\mathcal{E}(\lambda \cdot W^P), \ \lambda \text{ progressively measurable}, \ |\lambda| \leq h\Big\}.$$

Here $\mathcal{P}^{\mathsf{ngd}}(P)$ is the set of a-priori valuation measures equivalent to P which satisfy the no-good-deal restriction under P, but might fail to be local martingale measures for the stock price process S (yet they are with respect to the trivial market with only the riskless asset $S^0 \equiv 1$). In particular for each $P \in \mathcal{P}_H$, the set $\mathcal{P}^{\mathsf{ngd}}(P)$ is also m-stable and convex. This implies that the dynamic coherent risk measure $\rho^P : L^2(P) \to L^2(P, \mathcal{F}_t)$ is time-consistent (see e.g. Lemma 3.1 in Chapter 3) satisfying $\rho^P_\cdot(X) \geq \pi^{u,P}_\cdot(X)$ since $\mathcal{P}^{\mathsf{ngd}}(P) \supseteq \mathcal{Q}^{\mathsf{ngd}}(P)$. Furthermore from (4.21) and (4.25), we have for all $t \in [0,T]$ and $P \in \mathcal{P}_H$ that

$$\pi^u_t(X) = \underset{P'\in\mathcal{P}_H(t^+,P)}{\operatorname{ess\,sup}}^P \ \underset{\phi\in\Phi(P')}{\operatorname{ess\,inf}}^P \rho^{P'}_t\Big(X - \int_t^T \phi^{\mathsf{tr}}_s dB_s\Big), \ P\text{-a.s.}. \tag{4.26}$$

In addition for $X \in \mathbb{L}^2_H$ it can be inferred from [Bec09, Theorem 5.4] (see also Theorem 3.17 in Chapter 3) that there exists a family $\Big\{\bar{\phi}^P \in \Phi(P), \ P \in \mathcal{P}_H\Big\}$ of trading strategies satisfying

$$\pi^u_t(X) = \underset{P'\in\mathcal{P}_H(t^+,P)}{\operatorname{ess\,sup}}^P \rho^{P'}_t\Big(X - \int_t^T (\bar{\phi}^{P'}_s)^{\mathsf{tr}} dB_s\Big), \ P\text{-a.s.}, \quad \text{for all } t \in [0,T], \ P \in \mathcal{P}_H. \tag{4.27}$$

Moreover $\bar{\phi}^P$ is given for $P \in \mathcal{P}_H$ by

$$\widehat{a}^{1/2}_t \bar{\phi}^P_t = \widehat{\Pi}_t(\widehat{a}^{1/2}_t \mathcal{Z}^{P,X}_t), \ t \in [0,T], \ P\text{-a.s.}, \tag{4.28}$$

where $(\mathcal{Y}^{P,X}, \mathcal{Z}^{P,X}),\ P \in \mathcal{P}_H$, is the solution to the standard BSDE under P with terminal condition X and generator $-\widehat{F}_t(\cdot) = -F_t(\cdot, \widehat{a}_t),\ t \in [0,T]$, for F defined in (4.22), satisfying $\mathcal{Y}^{P,X} = \pi^{u,P}_\cdot(X)$. If $\mathcal{P}_H$ were a singleton $\mathcal{P}_H = \{P\}$, then for $X \in \mathbb{L}^2_H = L^2(P)$ the strategy $\bar{\phi}^P$ would be $\mathcal{P}_H$-permitted and hence already the solution to the good-deal hedging problem with the valuation $\pi^u_\cdot(X) = \pi^{u,P}_\cdot(X)$ associated to the risk measure ρ^P. In the present non-dominated framework however, the situation is more subtle because the strategies $\bar{\phi}^P$ and risk measures ρ^P may be defined only up to a null-set of the associated probability measure $P \in \mathcal{P}_H$. Since we are looking for a $\mathcal{P}_H$-permitted hedging strategy, one way is to investigate appropriate conditions under which the family $\{\bar{\phi}^P,\ P \in \mathcal{P}_H\}$ can be aggregated into a single strategy $\bar{\phi} \in \Phi$, i.e. $\bar{\phi} = \bar{\phi}^P\ P \otimes dt$-a.s., for any $P \in \mathcal{P}_H$. If this were possible, then (4.27) would write

$$\begin{aligned}
\pi^u_t(X) &= \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess sup}^P}\ \rho^{P'}_t\Big(X - \int_t^T \bar{\phi}^{\text{tr}}_s dB_s\Big),\ t \in [0,T],\ P\text{-a.s. for all } P \in \mathcal{P}_H \\
&= \rho_t\Big(X - \int_t^T \bar{\phi}^{\text{tr}}_s dB_s\Big),\ t \in [0,T],\ P\text{-a.s. for all } P \in \mathcal{P}_H,
\end{aligned}$$

where $\rho_\cdot(X) \in \mathbb{D}^2_H$ is defined for $X \in L^2_H$ as the unique process (if it exists) that satisfies

$$\rho_t(X) = \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess sup}^P}\ \rho^{P'}_t(X),\ t \in [0,T],\ P\text{-a.s. for all } P \in \mathcal{P}_H. \tag{4.29}$$

As general conditions for aggregation (see e.g. [STZ11]) can be somewhat restrictive and technical, we will express the hedging strategy in terms of the control component Z of the unique solution (Y,Z) to the 2BSDE (4.23). Note that even in case there exists a worst-case measure $\bar{P} \in \mathcal{P}_H$ such that $\rho = \rho^{\bar{P}}$, it is not not clear at all whether a hedging strategy in the model $\bar{P}$ is robust with respect to all measures in $\mathcal{P}^{\text{ngd}}(P)$ for any $P \in \mathcal{P}_H$, in the sense that the supermartingale property of tracking errors holds uniformly under any $Q \in \cup_{P \in \mathcal{P}_H} \mathcal{P}^{\text{ngd}}(P)$. An analogous issue was already noticed in Subsection 3.3.4 of Chapter 3 under drift uncertainty. The issue was addressed there by first considering a larger valuation bound for which a robust hedging strategy uniformly with respect to all priors exists, i.e. a strategy that satisfies a supermartingale property of tracking error under all measures a-priori valuation measure uniformly over all priors. A subsequent step was then to identify this larger bound with the standard good-deal valuation bound. Here relying on the intuition from Theorem 3.28 and Theorem 3.30 in Chapter 3, we can write down what a candidate hedging strategy (cf. (4.34)) in our setup in terms of the solution to the 2BSDE (4.23). From this we can then proceed in a more straightforward manner to show directly that this candidate strategy is indeed a good-deal hedging strategy and that it satisfies the required robustness property with respect to uncertainty.

Clearly ρ is a dynamic coherent risk measure analogous to $\pi^u_\cdot(X)$. The good-deal hedging problem under volatility uncertainty consists in minimizing over $\mathcal{P}_H$-permitted trading strategies the dynamic residual risk measured under ρ. This is done in such a way that at every time the minimal capital required for acceptability coincides with the good-deal valuation bound. More precisely for a contingent claim $X \in \mathcal{L}^2_H$, we aim to find $\bar{\phi} \in \Phi$ such that for all $t \in [0,T]$ and $P \in \mathcal{P}_H$ holds

$$\pi^u_t(X) = \underset{\phi\in\Phi}{\text{ess inf}}^P \; \rho_t\Big(X - \int_t^T \phi^{\mathsf{tr}}_s dB_s\Big) = \rho_t\Big(X - \int_t^T \bar{\phi}^{\mathsf{tr}}_s dB_s\Big), \; P\text{-a.s.}. \tag{4.30}$$

To introduce the notion of robustness with respect to volatility uncertainty, recall the definition of the tracking error $R^\phi(X)$ of a permitted strategy $\phi \in \Phi$ for a claim $X \in \mathbb{L}^2_H$:

$$R^\phi_t = \pi^u_t(X) - \pi^u_0(X) - \int_0^t \phi^{\mathsf{tr}}_s dB_s, \quad t \in [0,T], \; P\text{-a.s. for all } P \in \mathcal{P}_H. \tag{4.31}$$

In other words, the tracking error is the difference between the dynamic variations in the capital requirement and the profit or loss from trading. As in Subsection 3.3.3 of Chapter 3, we will say that a good-deal hedging strategy $\bar{\phi}(X)$ for a claim X is robust with respect to uncertainty if $R^{\bar{\phi}}(X)$ is a supermartingale under every measure $Q \in \mathcal{P}^{\mathsf{ngd}}(P)$ uniformly for all $P \in \mathcal{P}_H$. Again as in Chapter 3, this means that a robust hedging strategy $\bar{\phi}$ is at least mean-self-financing uniformly over all $Q \in \cup_{P\in\mathcal{P}_H}\mathcal{P}^{\mathsf{ngd}}(P)$.

Let us make a short transit and provide a 2BSDE description of the risk measure $\rho_\cdot(X)$, for $X \in \mathcal{L}^2_H$. As in part 4. of Remark 4.21, this yields in particular time-consistency of the dynamic risk measure ρ over contingent claims X in $\mathcal{L}^2_H$. For this purpose, define the function $F' : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ by

$$F'(t,z,a) = \begin{cases} -h_t|a^{1/2}z|, & \text{if } a \in \mathbb{S}^{>0}_n \cap [\underline{a}, \overline{a}], \\ +\infty & \text{otherwise.} \end{cases} \tag{4.32}$$

Consider the 2BSDE

$$Y'_t = X - \int_t^T Z'^{\mathsf{tr}}_s dB_s - \int_t^T \widehat{F}'_s(Z'_s)ds + K'_T - K'_t, \quad t \in [0,T], \quad \mathcal{P}_H\text{-q.s.}, \tag{4.33}$$

with generator F' defined in (4.32).

Proposition 4.23. *1. If $X \in \mathbb{L}^2_H$ and $(Y', Z') \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE (4.33), then $\rho_\cdot(X)$ is uniquely given by $\rho_t(X) = Y'_t,\ t \in [0,T],\ \mathcal{P}_H$-q.s. and satisfies*

$$\rho_t(X) = Y'_t = \underset{P'\in\mathcal{P}_H(t^+,P)}{\text{ess sup}^P} \; \underset{Q\in\mathcal{P}^{\mathsf{ngd}}(P')}{\text{ess sup}^{P'}} E^Q_t[X], \quad t \in [0,T], \; P\text{-a.s. for all } P \in \mathcal{P}_H.$$

2. For $X \in \mathcal{L}^2_H$, there exists a unique solution $(Y', Z') \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ to the 2BSDE (4.33).

Proof. Rewriting F' on $a \in \mathbb{S}_n^{>0} \cap [\underline{a}, \overline{a}]$ as $F'(t,z,a) = -h_t(z^{\mathrm{tr}}az)^{1/2}$ and using again the Schur complement condition (see proof of Theorem 4.20), one also proves that that F' is a convex function of a for fixed $(t,z) \in [0,T] \times \mathbb{R}^n$. This by Fenchel-Moreau theorem implies that F' is the convex conjugate of a nonlinear function H' such that an analog of (4.3) holds. In addition, it is easy to verify as in the proof of Theorem 4.20 that F' satisfies Assumption 4.6.

Part 1: By [Bec09], it is known that $\tilde{\mathcal{Y}}_t^{P,X} = \rho_t^P(X)$, P-a.s., $t \in [0,T]$, where $(\tilde{\mathcal{Y}}^{P,X}, \tilde{\mathcal{Z}}^{P,X})$ denotes the unique solution to the standard Lipschitz BSDE under P with generator $-\widehat{F}'$ and terminal condition X, for $P \in \mathcal{P}_H$. Hence part 1 is also a direct consequence of part 1 of Proposition 4.9 and the definition of ρ in (4.29).

Part 2: As a consequence of part 2 of Proposition 4.9, for $X \in \mathcal{L}_H^2$ the 2BSDE (4.33) admits a unique solution $(Y', Z') \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$. $\square$

We characterize $\bar{\phi}$ in terms of the unique solution (Y,Z) of the 2BSDE (4.23) and show that it is robust with respect to volatility uncertainty. Using the intuition from robust hedging in the presence of drift uncertainty (see Theorem 3.28 and Theorem 3.30 in Chapter 3), a candidate good-deal hedging strategy for $X \in \mathcal{L}_H^2$ is $\bar{\phi} := \bar{\phi}(X)$ defined by

$$\widehat{a}_t^{1/2}\bar{\phi}_t := \widehat{\Pi}_t(\widehat{a}_t^{1/2}Z_t), \quad t \in [0,T],\ \mathcal{P}_H\text{-q.s.}, \tag{4.34}$$

where (Y,Z) is a solution to the 2BSDE (4.23). Since Z is already defined $\mathcal{P}_H$-quasi-surely and $\widehat{a}$ is defined pathwise, then the strategy $\bar{\phi}$ in (4.34) is also defined $\mathcal{P}_H$-quasi-surely and it can be shown that it is indeed a robust good-deal hedging strategy if the "gain/loss" family of processes $\left\{{}^{(P)}\!\int_0^{\cdot} Z_t^{\mathrm{tr}} dB_t,\ P \in \mathcal{P}_H\right\}$ aggregates. The precise result is the following

Theorem 4.24. *Assume $X \in \mathbb{L}_H^2$ and that $(Y,Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is a solution to the 2BSDE (4.23) with generator F given by (4.22) such that the integrals $\left\{{}^{(P)}\!\int_0^{\cdot} Z_t^{\mathrm{tr}} dB_t,\ P \in \mathcal{P}_H\right\}$, aggregate into a single process $\int_0^{\cdot} Z_t^{\mathrm{tr}} dB_t$ (equivalently 2BSDE (4.23) admits a solution (Y,Z,K)). Then:*

1. *The strategy $\bar{\phi} = \bar{\phi}(X)$ given by (4.34) is in Φ and solves the good-deal hedging problem under uncertainty (4.30).*
2. *The tracking error process $R^{\bar{\phi}}(X)$ of the hedging strategy $\bar{\phi} = \bar{\phi}(X)$ is a supermartingale under any Q in $\cup_{P \in \mathcal{P}_H} \mathcal{P}^{\mathrm{ngd}}(P)$.*

Proof. We first prove part 2., since the proof of part 1. will use it. By Theorem 4.20, we know that $\pi_\cdot^u(X) = Y$ for (Y,Z) solution to the 2BSDE (4.23). Let $P \in \mathcal{P}_H$ and $Q \in \mathcal{P}^{\mathrm{ngd}}(P)$. Then Q is equivalent to P and $dQ = {}^{(P)}\mathcal{E}(\lambda \cdot W^P)dP$ for $|\lambda| \le h$. The dynamics of $R^{\bar{\phi}} := R^{\bar{\phi}}(X)$

is then given under P by

$$\begin{aligned}-dR^{\bar\phi}_t &= -\widehat{F}_t(Z_t)dt - Z_t^{\mathrm{tr}}dB_t + \bar\phi_t^{\mathrm{tr}}dB_t + dK^P_t, \quad P\text{-a.s.},\\ &= -\widehat{F}_t(Z_t)dt - (Z_t-\bar\phi_t)^{\mathrm{tr}}\widehat{a}_t^{\frac12}dW^P_t + dK^P_t, \quad P\text{-a.s.},\end{aligned}$$

for all $t\in[0,T]$, with $\{K^P,\ P\in\mathcal{P}_H\}$ the non-decreasing adapted processes defined as in (4.7). Changing measures to Q for the Q-Brownian motion $W^Q = W^P - \int_0^\cdot \lambda_t dt$ gives for $t\in[0,T]$ that

$$-dR^{\bar\phi}_t = (-\widehat{F}_t(Z_t) - \lambda_t^{\mathrm{tr}}\widehat{a}_t^{\frac12}(Z_t-\bar\phi_t))dt - (Z_t-\bar\phi_t)^{\mathrm{tr}}\widehat{a}_t^{\frac12}dW^Q_t + dK^P_t, \quad P\text{-a.s.}$$

holds. Now with (4.22) and the expression of $\bar\phi$ in (4.34) one rewrites P-a.s. for $t\in[0,T]$

$$-dR^{\bar\phi}_t = \Big(h_t\big|\widehat{\Pi}^\perp_t(\widehat{a}_t^{\frac12}Z_t)\big| - \lambda_t^{\mathrm{tr}}\widehat{\Pi}^\perp_t(\widehat{a}_t^{\frac12}Z_t)\Big)dt - (Z_t-\bar\phi_t)^{\mathrm{tr}}\widehat{a}_t^{\frac12}dW^Q_t + dK^P_t.$$

Since K^P is non-decreasing and $\max_{|\lambda_t|\le h_t}\lambda_t^{\mathrm{tr}}\widehat{\Pi}^\perp_t(\widehat{a}_t^{\frac12}Z_t) = h_t\big|\widehat{\Pi}^\perp_t(\widehat{a}_t^{\frac12}Z_t)\big|$, then the finite variation part of the Q-semimartingale $R^{\bar\phi}$ is non-increasing. Note that $R^{\bar\phi}\in\mathcal{S}^2(P)$ since $\pi^u_\cdot(X)\in\mathbb{D}^2_H\subset\mathcal{S}^2(P)$ and $\bar\phi\in\Phi(P)$. Finally, since λ is bounded, $\frac{dQ}{dP}$ is in $L^p(P)$ for any $p<\infty$ and by Hölder's inequality it follows that $R^{\bar\phi}\in\mathcal{S}^{2-\varepsilon}(Q)$ $(\varepsilon>0)$ holds. As a consequence $R^{\bar\phi}$ is clearly a Q-supermartingale.

Now to prove part 1. note first that by the condition on the integral Z, the strategy $\bar\phi$ given by (4.34) belongs to Φ. Now to show that $\bar\phi$ solves the hedging problem (4.30), let $P\in\mathcal{P}_H$, and $P'\in\mathcal{P}_H(t^+,P)$. Then for any $\phi\in\Phi$ it holds $\phi\cdot B$ is a Q-martingale in $\mathcal{S}^1(Q)$ for any $Q\in\mathcal{Q}^{\mathrm{ngd}}(P')$ since the Girsanov kernels of measures Q with respect to P' are all uniformly bounded. Because $\mathcal{Q}^{\mathrm{ngd}}(P')\subseteq\mathcal{P}^{\mathrm{ngd}}(P')$, this implies that $\pi^{u,P'}_t(X) = \pi^{u,P'}_t(X-\int_t^T\phi_s^{\mathrm{tr}}dB_s)\le \rho^{P'}_t(X-\int_t^T\phi_s^{\mathrm{tr}}dB_s)$, P'-a.s.. Taking the essential supremum over $P'\in\mathcal{P}_H(t^+,P)$ first and then the essential infimum over $\phi\in\Phi$ yields $\pi^u_t(X)\le \text{ess inf}^P_{\phi\in\Phi}\rho_t(X-\int_t^T\phi_s^{\mathrm{tr}}dB_s)$, P-a.s.. Hence to show that $\bar\phi$ is a good-deal hedging strategy satisfying (4.30), it suffices to show that $\pi^u_t(X)\ge E^Q_t[X-\int_t^T\bar\phi_s^{\mathrm{tr}}dB_s]$, P-a.s. holds for all $Q\in\mathcal{P}^{\mathrm{ngd}}(P')$ and $P'\in\mathcal{P}_H(t^+,P)$. To this end, let $P'\in\mathcal{P}_H(t^+,P)$ and $Q\in\mathcal{P}^{\mathrm{ngd}}(P')$. From part 1. of the theorem, the supermartingale property of the tracking error $R^{\bar\phi} := R^{\bar\phi}_\cdot(X)$ of $\bar\phi$ under Q implies that $\pi^u_t(X)-\pi^u_0(X)-\int_0^t\bar\phi_s^{\mathrm{tr}}dB_s\ge E^Q_t[X-\pi^u_0(X)-\int_0^T\bar\phi_s^{\mathrm{tr}}dB_s]$. Reorganizing the last inequality yields the claim. □

In general, the Itô's stochastic integrals of the form $\int_0^\cdot Z_t^{\mathrm{tr}}dB_t$ are only defined P-almost surely under each $P\in\mathcal{P}_H$. As already mentioned before, sufficient conditions for aggregation of processes can be quite restrictive. By a result of [Kar95] it is possible to define $\int_0^\cdot Z_t^{\mathrm{tr}}dB_t$ pathwise, and in particular such that it satisfies the hypothesis of Theorem 4.24, if the process Z is càdlàg. Note that in our setup the Z-component of a 2BSDE solution (Y,Z) does not have to be a càdlàg process in general. We emphasize however that this does not make Theorem 4.24 totally inapplicable. Indeed under some Markovian assumptions one may sometimes be able to use PDE arguments to show that the Z-component is càdlàg. An example of such a situation is provided in Section 4.3.3 below, where we obtain explicit solutions to the 2BSDE (4.23), for some contingent claim satisfying $X \in \mathbb{L}^2_H$ and probably not $X \in \mathcal{L}^2_H$. In a general context, a result of [Nut12b] shows that the stochastic integral $\int_0^\cdot Z_t^{\mathrm{tr}} dB_t$ can be defined pathwise for any predictable process Z if one complements the Zermelo-Fraenkel axioms of set theory (which are by now well-accepted) with the combination "continuum hypothesis plus the axiom of choice" or the softer one "negation of the continuum hypothesis plus the so-called Martin's axiom" (see [DM78, Chapter II, Sections 27-29]). Note under the conditions of [Nut12b] that for a solution (Y, Z) of a 2BSDE for which Y is $\mathcal{P}_H$-quasi-surely defined, the family $\{K^P,\, P \in \mathcal{P}_H\}$ will automatically aggregate into a single process K such that (Y, Z, K) becomes a solution to the 2BSDE. As for the 2BSDE (4.23) of interest in Theorem 4.24, we already know by Theorem 4.20 that this would be the case if $X \in \mathcal{L}^2_H$.

As a further remark, note that Part 2. of Theorem 4.24 can be interpreted as a robustness property of the good-deal hedging strategy $\bar{\phi}$ with respect to volatility uncertainty. Finally, a direct consequence of Theorem 4.24 (when its conditions are satisfied) is the following minmax identity: for all $t \in [0, T]$, $P \in \mathcal{P}_H$ one has by (4.26) and (4.30) that

$$\begin{aligned}\bar{\pi}_t^u(X) :=&\ \underset{\phi\in\Phi}{\operatorname{ess\,inf}^P}\ \underset{P'\in\mathcal{P}_H(t^+,P)}{\operatorname{ess\,sup}^P}\ \rho_t^{P'}\Big(X - \int_t^T \phi_s^{\mathrm{tr}} dB_s\Big)\\ =&\ \underset{P'\in\mathcal{P}_H(t^+,P)}{\operatorname{ess\,sup}^P}\ \underset{\phi\in\Phi(P')}{\operatorname{ess\,inf}^P}\ \rho_t^{P'}\Big(X - \int_t^T \phi_s^{\mathrm{tr}} dB_s\Big) = \pi_t^u(X),\ P\text{-a.s.,}\end{aligned}$$

4.3.3 Example for options on non-traded assets

We provide an example for robust good-deal valuation and hedging of European put options on a non-traded asset under volatility uncertainty. The financial market consists of a traded stock of Black-Scholes' type with (discounted) price process S and a non-traded asset with value process L. Hence $d = 1$ and $n = 2$ for the framework of Section 4.2. For the canonical process $B = (B^1, B^2)$, the set $\mathcal{P}_H$ of local martingale measures is defined as in (4.1) via constant diagonal matrices $\underline{a}, \overline{a} \in \mathbb{S}_2^{>0}$ given by $\underline{a} = \mathrm{diag}(\underline{a}_1, \underline{a}_2)$ and $\overline{a} = \mathrm{diag}(\overline{a}_1, \overline{a}_2)$, such that $\underline{a} \le \widehat{a} \le \overline{a}$, $\mathcal{P}_H \otimes dt$-q.s.. We model (S, L) as

$$dS_t = S_t \sigma^S dB_t^1 \quad \text{and} \quad dL_t = L_t\big(\gamma dt + \beta(\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2)\big), \quad \mathcal{P}_H\text{-q.s.,}$$

with $S_0, L_0 > 0$, a volatility matrix $\sigma := (\sigma^S, 0) \in \mathbb{R}^{1\times 2}$ of maximal rank $1 = d < n = 2$, $\sigma^S, \beta \in (0, \infty)$, $\gamma \in \mathbb{R}$, and P^0-correlation coefficient $\rho \in [-1, 1]$ for returns of S and L. For a constant bound $h \in [0, \infty)$ on the instantaneous Sharpe ratios, we derive explicit formulas for the worst-case good-deal valuation bound and robust hedging strategy of European put options $X = (\mathcal{K} - L_T)^+$, with strike $\mathcal{K} \in \mathbb{R}_+$ and maturity T. We denote

$$\widehat{a} = \begin{pmatrix} \widehat{a}^{11} & \widehat{a}^{12} \\ \widehat{a}^{12} & \widehat{a}^{22} \end{pmatrix} \quad \text{and} \quad \widehat{a}^{\frac{1}{2}} = \begin{pmatrix} \widehat{b}^{11} & \widehat{b}^{12} \\ \widehat{b}^{12} & \widehat{b}^{22} \end{pmatrix},$$

for $\widehat{a}$ being the $\mathbb{S}_2^{>0}$-valued process satisfying $d\langle B\rangle_t = \widehat{a}_t dt$ pathwise and $\underline{a} \leq \widehat{a} \leq \overline{a}$, $\mathcal{P}_H \otimes dt$-q.s. as in (4.1). One has

$$\begin{aligned} \widehat{a}^{11} = (\widehat{b}^{11})^2 + (\widehat{b}^{12})^2, \quad \widehat{a}^{12} = \widehat{b}^{12}(\widehat{b}^{11} + \widehat{b}^{22}), \quad \widehat{a}^{22} = (\widehat{b}^{22})^2 + (\widehat{b}^{12})^2, \\ \text{and} \quad \widehat{a}^{11}\widehat{a}^{22} - (\widehat{a}^{12})^2 = \big(\widehat{b}^{11}\widehat{b}^{22} - (\widehat{b}^{12})^2\big)^2. \end{aligned} \tag{4.35}$$

Since $\sigma\widehat{a}^{\frac{1}{2}} = \sigma^S(\widehat{b}^{11}, \widehat{b}^{12})$, then from their respective definitions hold

$$\mathrm{Im}\,(\sigma\widehat{a}^{\frac{1}{2}})^{\mathsf{tr}} = \{z \in \mathbb{R}^2 : \widehat{b}^{12}z_1 - \widehat{b}^{11}z_2 = 0\} \quad \text{and} \quad \mathrm{Ker}\,(\sigma\widehat{a}^{\frac{1}{2}}) = \{z \in \mathbb{R}^2 : \widehat{b}^{11}z_1 + \widehat{b}^{12}z_2 = 0\},$$

which imply for $z \in \mathbb{R}^2$ that

$$\widehat{\Pi}(z) = \frac{1}{\widehat{a}^{11}} \begin{pmatrix} (\widehat{b}^{11})^2 z_1 + \widehat{b}^{11}\widehat{b}^{12} z_2 \\ \widehat{b}^{11}\widehat{b}^{12} z_1 + (\widehat{b}^{12})^2 z_2 \end{pmatrix} \text{ and } \widehat{\Pi}^{\perp}(z) = \frac{1}{\widehat{a}^{11}} \begin{pmatrix} (\widehat{b}^{12})^2 z_1 - \widehat{b}^{11}\widehat{b}^{12} z_2 \\ (\widehat{b}^{11})^2 z_2 - \widehat{b}^{11}\widehat{b}^{12} z_1 \end{pmatrix}. \tag{4.36}$$

Clearly $L_T \in \mathbb{L}_H^2$ follows from the estimate (4.44) below, since the process $\tilde{L}$ there satisfies $E_t^P[\tilde{L}] \leq 1$ for any $t \in [0,T]$, $P \in \mathcal{P}_H$. In addition since the put option payoff function $x \mapsto (\mathcal{K} - x)^+$ is bounded and Lipschitz continuous, it follows that $X = (\mathcal{K} - L_T)^+ \in \mathbb{L}_H^2$. Recall from (4.21) that the worst-case good-deal bound $\pi_t^u(X)$ (if it exists) for X for $t \in [0,T]$ satisfies for any $P \in \mathcal{P}_H$

$$\pi_t^u(X) = \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess sup}^P} \pi_t^{u,P'}(X) = \underset{P' \in \mathcal{P}_H(t^+,P)}{\text{ess sup}^P} \underset{Q \in \mathcal{Q}^{\mathsf{ngd}}(P')}{\text{ess sup}^P} E_t^Q[X], \ P\text{-a.s.}.$$

From (4.36) and using (4.35), follows $\big|\widehat{\Pi}^{\perp}(\widehat{a}^{1/2}z)\big| = \big(\widehat{a}^{11}\widehat{a}^{22} - (\widehat{a}^{12})^2\big)^{1/2}\big(\widehat{a}^{11}\big)^{-1/2}|z_2|$. is a solution to the 2BSDE (4.23) which rewrites here as

$$Y_t = X - \int_t^T \widehat{F}(s, Z_s)ds - \int_t^T Z_s^{\mathsf{tr}} dB_s + K_T - K_t, \ t \in [0,T], \ \mathcal{P}_H\text{-q.s.}, \tag{4.37}$$

with generator given by $F(t,z,a) = -h\big|\Pi^{a,\perp}(a^{1/2}z)\big| = -h\big(a^{11}a^{22} - (a^{12})^2\big)^{1/2}\big(a^{11}\big)^{-1/2}|z_2|$, for $a \in \mathbb{S}_2^{>0} \cap [\underline{a}, \overline{a}]$ and $F(t,z,a) = +\infty$ otherwise, for $z = (z_1, z_2)^{\mathsf{tr}} \in \mathbb{R}^2$. We show in Lemma 4.25 below that the solution to the BSDE (4.37) is given by

$$Y_t = v(t, L_t) \quad \text{and} \quad Z_t = \beta L_t \frac{\partial v}{\partial x}(t, L_t)\big(\rho, \sqrt{1-\rho^2}\big)^{\mathsf{tr}}, \ \mathcal{P}_H\text{-q.s.}$$

hold for every $t \in [0,T]$, where $v \in \mathcal{C}^{1,2}\big([0,T) \times (0,\infty), \mathbb{R}\big)$ is the classical solution to the Black-Scholes' type PDE

$$\begin{cases} \frac{\partial v}{\partial t} + \big(\gamma - h\beta\sqrt{1-\rho^2}\sqrt{\overline{a}_2}\big)\, x\, \frac{\partial v}{\partial x} + \frac{1}{2}\beta^2\left(\rho^2\overline{a}_1 + (1-\rho^2)\overline{a}_2\right) x^2 \frac{\partial^2 v}{\partial x^2} = 0 \\ v(T, L_T) = (\mathcal{K} - L_T)^+. \end{cases} \tag{4.38}$$

We need some preparations towards this result. Let $P^{\overline{a}} = P^0 \circ (\overline{a}^{1/2}B)^{-1} \in \mathcal{P}_H$ be the local martingale measure satisfying $\langle B\rangle_t = \overline{a}t,\ P^{\overline{a}}$-a.s., for all $t \in [0,T]$. The dynamics of the process L under $P^{\overline{a}}$ is the geometric Brownian motion

$$dL_t = L_t\Big(\gamma dt + \beta(\rho\sqrt{\overline{a}_1}dW_t^{1,P^{\overline{a}}} + \sqrt{1-\rho^2}\,\sqrt{\overline{a}_2}dW_t^{2,P^{\overline{a}}})\Big),\ t\in[0,T],$$

which can be rewritten as $dL_t = L_t\Big(\gamma dt + \bar{\beta}(\bar{\rho}dW_t^{1,P^{\overline{a}}} + \sqrt{1-\bar{\rho}^2}\,dW_t^{2,P^{\overline{a}}})\Big),\ t\in[0,T]$, for

$$\bar{\beta} := \beta\Big(\rho^2\overline{a}_1 + (1-\rho^2)\overline{a}_2\Big)^{1/2} > 0 \quad \text{and} \quad \bar{\rho} := \rho\sqrt{\overline{a}_1}\Big(\rho^2\overline{a}_1 + (1-\rho^2)\overline{a}_2\Big)^{-1/2} \in [-1,1],$$

where $W^{P^{\overline{a}}} = (W^{1,P^{\overline{a}}}, W^{2,P^{\overline{a}}}) = (\overline{a})^{-1/2}B$ is a $P^{\overline{a}}$-Brownian motion. The Black-Scholes formula applied for the dynamics of L under $P^{\overline{a}}$ provides a closed-form expression for $v(t,L_t)$, for v solution to the PDE (4.38). Using arguments analogous to the ones in the derivations of (3.31) in Section 3.2.2 of Chapter 3 it can be shown that $v(t,L_t)$ coincides with the good-deal valuation bound $\pi_t^{u,P^{\overline{a}}}((\mathcal{K}-L_T)^+)$ in the model under $P^{\overline{a}}$. Furthermore, an explicit formula for both is given by $\mathcal{K}$

$$\begin{aligned} v(t,L_t) &= \pi_t^{u,P^{\overline{a}}}(X) \\ &= \mathcal{K}N(-d_-) - L_t e^{m(T-t)}N(-d_+) \\ &= e^{m(T-t)} * \text{B/S-put-price(time: } t,\ \text{spot price: } L_t,\ \text{strike: } \mathcal{K}e^{-m(T-t)}, \text{vol: } \bar{\beta}), \end{aligned} \tag{4.39}$$

with " B/S-put-price" being the standard Black-Scholes formula for interest rate being zero, with "vol" being the argument for volatility in the Black-Scholes model, where

$$m := \gamma - h\bar{\beta}\sqrt{1-\bar{\rho}^2} = \gamma - h\beta\sqrt{1-\rho^2}\sqrt{\overline{a}_2},$$

$d_\pm := \big(\ln(L_t/\mathcal{K}) + (m \pm \frac{1}{2}\bar{\beta}^2)(T-t)\big)\big(\bar{\beta}\sqrt{(T-t)}\big)^{-1}$ and N is the cumulative distribution function of the standard normal law. The following lemma identifies the solution to the 2BSDE (4.37) via the solution v of the PDE (4.38).

Lemma 4.25. *The solution (Y,Z,K) of the 2BSDE (4.37) is given by $Y_t = v(t,L_t)$, $Z_t = \beta L_t \frac{\partial v}{\partial x}(t,L_t)(\rho,\sqrt{1-\rho^2})^{\mathrm{tr}}$ and K given by (4.41), for $t\in[0,T]$, with $(Y,Z)\in\mathbb{D}_H^2\times\mathbb{H}_H^2$ such that the stochastic integral $\int_0^\cdot Z_t^{\mathrm{tr}}dB_t$ is pathwise defined.*

Proof. For any $P\in\mathcal{P}_H$, applying Itô's formula and using (4.38) yields for $t\in[0,T]$

$$v(t,L_t) = X - \int_t^T Z_s^{\mathrm{tr}}dB_s + h\int_t^T \big(\widehat{a}_s^{11}\widehat{a}_s^{22} - (\widehat{a}_s^{12})^2\big)^{1/2}\big(\widehat{a}_s^{11}\big)^{-1/2}|Z_s^2|ds + K_T - K_t,\ P\text{-a.s.},$$

with $Z = (Z^1,Z^2)^{\mathrm{tr}}$ given from (4.39) by

$$Z_t = \beta L_t\frac{\partial v}{\partial x}(t,L_t)(\rho,\sqrt{1-\rho^2})^{\mathrm{tr}} = -\beta e^{m(T-t)}N(-d_+)L_t(\rho,\sqrt{1-\rho^2})^{\mathrm{tr}} \tag{4.40}$$

and

$$K_t := \int_0^t \Big[h\beta\sqrt{1-\rho^2} L_s \frac{\partial v}{\partial x}(s, L_s)\Big(\big(\widehat{a}_s^{11}\widehat{a}_s^{22} - (\widehat{a}_s^{12})^2\big)^{1/2} \big(\widehat{a}_s^{11}\big)^{-1/2} - \sqrt{\overline{a}_2}\Big) \\ + \frac{1}{2}\beta^2 L_s^2 \frac{\partial^2 v}{\partial x^2}(s, L_s)\Big(\rho^2(\overline{a}_1 - \widehat{a}_s^{11}) + (1-\rho^2)(\overline{a}_2 - \widehat{a}_s^{22}) - 2\rho\sqrt{1-\rho}\widehat{a}_s^{12}\Big)\Big] ds. \tag{4.41}$$

To show that K is a non-decreasing process, note that $\widehat{a} \le \overline{a}$ $P \otimes dt$-a.s. yields $\widehat{a}^{1/2} \le \overline{a}^{1/2}$ $P \otimes dt$-a.s. and both inequalities imply that

$$\rho^2(\overline{a}_1 - \widehat{a}^{11}) + (1-\rho^2)(\overline{a}_2 - \widehat{a}^{22}) - 2\rho\sqrt{1-\rho}\,\widehat{a}^{12} \\ = (\rho, \sqrt{1-\rho^2})^{\mathrm{tr}}\overline{a}\,(\rho, \sqrt{1-\rho^2}) - (\rho, \sqrt{1-\rho^2})^{\mathrm{tr}}\widehat{a}\,(\rho, \sqrt{1-\rho^2}) \ge 0 \tag{4.42}$$

and

$$\big(\widehat{a}^{11}\widehat{a}^{22} - (\widehat{a}^{12})^2\big)^{1/2}\big(\widehat{a}^{11}\big)^{-1/2} \le \big(\widehat{a}^{22}\big)^{1/2} \le \sqrt{\overline{a}_2} \tag{4.43}$$

hold $P \otimes dt$-almost surely. Thus the process K is P-a.s. non-decreasing, because the delta of the put option in the Black-Scholes model is non-positive and the gamma is non-negative, i.e. $\frac{\partial v}{\partial x}(t, L_t) \le 0$ and $\frac{\partial^2 v}{\partial x^2}(t, L_t) \ge 0$ for all $t \in [0, T]$ using (4.39). Moreover the process K satisfies the minimum condition (4.8). This can be shown following arguments analogous to those in the proof of [STZ12, Theorem 5.3]; we reproduce the arguments for the convenience of the reader. Indeed, let us define $l : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ basically as the generator function (minus the γ-term) of the PDE (4.38) defined for $(x, p, q) \in \mathbb{R}^+ \times \mathbb{R}^2$ by

$$l(x, p, q) := -h\beta\sqrt{1-\rho^2}\sqrt{\overline{a}_2}\, xp + \frac{1}{2}\beta^2\left(\rho^2\overline{a}_1 + (1-\rho^2)\overline{a}_2\right)x^2 q,$$

so that $K = \int_0^\cdot k_s ds$ holds with

$$k_t = l\big(L_t, \frac{\partial v}{\partial x}(t, L_t), \Gamma_t\big) - \frac{1}{2}\beta^2 L_t^2\Big(\rho^2\widehat{a}_t^{11} + (1-\rho^2)\widehat{a}_t^{22} + 2\rho\sqrt{1-\rho}\widehat{a}_t^{12}\Big)\Gamma_t + \widehat{F}(t, Z_t),$$

for $\Gamma_t = \frac{\partial^2 v}{\partial x^2}(t, L_t)$ and Z_t given by (4.40). Since $l\big(L_t, \frac{\partial v}{\partial x}(t, L_t), \Gamma_t\big)$ is by (4.42) and (4.43) the supremum of $\frac{1}{2}\beta^2 L_t^2\Big(\rho^2 a_t^{11} + (1-\rho^2)a_t^{22} + 2\rho\sqrt{1-\rho}a_t^{12}\Big)\Gamma_t - \widehat{F}(t, Z_t)$ over $a \in D_F := [\underline{a}, \overline{a}]$, then by measurable selection arguments there exists for every $\epsilon > 0$ a predictable process a^ϵ valued in D_F such that

$$l\big(L_t, \frac{\partial v}{\partial x}(t, L_t), \Gamma_t\big) \le \frac{1}{2}\beta^2 L_t^2\Big(\rho^2 a_t^{\epsilon,11} + (1-\rho^2)a_t^{\epsilon,22} + 2\rho\sqrt{1-\rho}a_t^{\epsilon,12}\Big)\Gamma_t \\ - F(t, Z_t, a_t^\epsilon) + \epsilon.$$

Now let $P^\alpha \in \mathcal{P}_H$ and $t_0 \in [0, T]$ be fixed, and define recursively the sequence $(\tau_n)_n$ of random times $\tau_0^\epsilon := \inf\{t \ge t_0 : K_t \ge K_{t_0} + \epsilon\} \wedge T$, and

$$\tau_{n+1}^\epsilon := \inf\Big\{t \ge \tau_n^\epsilon : \quad l\big(L_t, \frac{\partial v}{\partial x}(t, L_t), \Gamma_t\big) + F(t, Z_t, a_{\tau_n^\epsilon}^\epsilon) \ge \\ \frac{1}{2}\beta^2 L_t^2\Big(\rho^2 a_{\tau_n^\epsilon}^{\epsilon,11} + (1-\rho^2)a_{\tau_n^\epsilon}^{\epsilon,22} + 2\rho\sqrt{1-\rho}a_{\tau_n^\epsilon}^{\epsilon,12}\Big)\Gamma_t + 2\epsilon\Big\} \wedge T.$$

Since K, L, Z, Γ are continuous, then τ_n^ϵ is a $\mathbb{F}$-stopping time for any n, and $\tau_0^\epsilon > t_0$. Furthermore since l is continuous and $F(\cdot, a)$ also is for fixed a in D_F, then for $\mathcal{P}_H$-quasi all ω the function

$$l(L_t, \frac{\partial v}{\partial x}(t, L_t), \Gamma_t) - \frac{1}{2}\beta^2 L_t^2\Big(\rho^2 a^{\epsilon,11}_{\tau_n^\epsilon} + (1-\rho^2)a^{\epsilon,22}_{\tau_n^\epsilon} + 2\rho\sqrt{1-\rho}a^{\epsilon,12}_{\tau_n^\epsilon}\Big)\Gamma_t + F(t, Z_t, a^\epsilon_{\tau_n^\epsilon})$$

is uniformly continuous in t on the compact interval $[\tau_n^\epsilon, T]$. Hence uniformly over n holds $\tau_{n+1}^\epsilon(\omega) - \tau_n^\epsilon(\omega) \geq \delta(\epsilon, \omega) > 0$ whenever $\{\tau_n^\epsilon(\omega) < T\}$, which implies $\tau_n^\epsilon(\omega) = T$ for large enough n. Now from the arguments in [STZ11, Example 4.5] applied to the interval $[\tau_0^\epsilon, T]$, there exists a $\mathbb{F}$-progressively measurable process α^ϵ valued in D_F such that

$$\alpha^\epsilon = \alpha \text{ on } [0, \tau_0^\epsilon] \quad \text{and} \quad \widehat{a} = \sum_{n=0}^{\infty} a^\epsilon_{\tau_n^\epsilon} \mathbb{1}_{[\tau_n^\epsilon, \tau_{n+1}^\epsilon)},\ P^{\alpha^\epsilon} \otimes dt\text{-a.s. on } \Omega \times [\tau_0^\epsilon, T].$$

It follows that $k \leq 2\epsilon$, $P^{\alpha^\epsilon} \otimes dt$-a.s. on $\Omega \times [\tau_0^\epsilon, T]$, which implies for $P := P^\alpha \in \mathcal{P}_H$ that

$$0 \leq \underset{P' \in \mathcal{P}_H(t_0+, P)}{\overset{P}{\text{ess inf}}}\, E^{P'}_{t_0}[K_T - K_{t_0}] \leq \epsilon + E^{P^{\alpha^\epsilon}}_{t_0}[K_T - K_{\tau_0^\epsilon}] \leq \epsilon + 2\epsilon(T - t_0),\ P\text{-a.s.},$$

since $P^{\alpha^\epsilon} \in \mathcal{P}(t_0+, P)$ because $\tau_0^\epsilon > t_0$. Taking the limit as ϵ tends to zero yields that K satisfies the minimum condition (4.8).

It remains to show that $v(\cdot, L.) \in \mathbb{D}^2_H$ and $Z \in \mathbb{H}^2_H$. This will conclude by uniqueness of the solution to the 2BSDE (4.23) (see Theorem 4.20) that $(v(\cdot, L.), Z)$ for Z given in (4.40) is the unique solution to the 2BSDE (4.37). Since v is of class $\mathcal{C}^{1,2}$ and L is $\mathcal{P}_H$-q.s. continuous, then $v(\cdot, L.)$ and Z are $\mathbb{F}^+$-progressively measurable. That $v(\cdot, L.)$ is in $\mathbb{D}^2_H$ now follows from (4.39) which indeed implies that $0 \leq v(t, L_t) \leq \mathcal{K}$ holds pathwise. From (4.40) and since $\underline{a} \leq \widehat{a} \leq \overline{a}$ holds P-a.s. for any $P \in \mathcal{P}_H$, one has $\left|\widehat{a}_t^{1/2} Z_t\right|^2 \leq \max(\overline{a}_1, \overline{a}_2)\beta^2 e^{2|m|T} L_t^2$ for all $t \in [0, T]$, P-a.s. for any $P \in \mathcal{P}_H$. Hence to show $Z \in \mathbb{H}^2_H$ it suffices to show that $\sup_{P \in \mathcal{P}_H} E^P[\int_0^T L_t^2 dt] < \infty$. For this purpose, note that for any $P \in \mathcal{P}_H$ it holds that

$$\int_0^T L_t^2 dt \leq \beta^{-2}(\min(\underline{a}_1, \underline{a}_2))^{-1}\langle L\rangle_T \quad \text{and} \quad L_T^2 \leq L_0^2 e^{\left(2|\gamma| + \beta^2 \max(\overline{a}_1, \overline{a}_2)\right)T} \tilde{L}_T \tag{4.44}$$

P-almost surely, for $\tilde{L}$ satisfying $\tilde{L} = 1 + \int_0^\cdot 2\tilde{L}_s\beta(\rho B^1_s + \sqrt{1-\rho^2}B^2_s)$, $\mathcal{P}_H$-q.s.. Clearly $E^P[\tilde{L}_T] \leq 1$ holds for every $P \in \mathcal{P}_H$, and thus taking expectations in (4.44) gives

$$E^P[\int_0^T L_t^2 dt] \leq \beta^{-2}(\min(\underline{a}_1, \underline{a}_2))^{-1} L_0^2 e^{\left(2|\gamma| + \beta^2 \max(\overline{a}_1, \overline{a}_2)\right)T}, \quad \text{for all } P \in \mathcal{P}_H.$$

Now taking the supremum over $P \in \mathcal{P}_H$ implies the result. So $(v(\cdot, L.), Z)$ is the unique solution to the 2BSDE (4.37) in $\mathbb{D}^2_H \times \mathbb{H}^2_H$. Finally that $\int_0^\cdot Z_t^{\text{tr}} dB_t$ is pathwise defined follows from [Kar95] since Z is continuous and $\mathbb{F}^+$-adapted. □

Lemma 4.25 implies that

$$\pi_t^u\big((\mathcal{K}-L_T)^+\big)=v(t,L_t)\quad\text{and}\quad Z_t=(Z_t^1,Z_t^2)^{\mathsf{tr}}=\beta L_t\frac{\partial v}{\partial x}(t,L_t)\Big(\rho,\sqrt{1-\rho^2}\Big)^{\mathsf{tr}},\ t\in[0,T].$$

Hence the robust good-deal bound $\pi_t^u\big((\mathcal{K}-L_T)^+\big)$ is attained for the largest "volatility matrix" $\overline{a}$, and can be computed as in the absence of uncertainty, but under a worst-case measure $P^{\overline{a}}\in\mathcal{P}_H$ for which $\langle B\rangle_t=\overline{a}t$ $P^{\overline{a}}$-a.s., yielding $\pi_t^u\big((\mathcal{K}-L_T)^+\big)=\pi_t^{u,P^{\overline{a}}}\big((\mathcal{K}-L_T)^+\big)$ for $t\in[0,T]$. In addition, $\pi_t^u\big((\mathcal{K}-L_T)^+\big)$ is given explicitly by the Black-Scholes type formula (4.39), for modified strike price and volatility corresponding to $\mathcal{K}\exp(-m(T-t))$ and $\bar{\beta}=\beta\big(\rho^2\overline{a}_1+(1-\rho^2)\overline{a}_2\big)^{1/2}$ respectively. Similarly, one can show that the lower good-deal bound $\pi_t^l\big((\mathcal{K}-L_T)^+\big)$ can be computed as in the absence of uncertainty, but under the worst-case measure $P^{\underline{a}}\in\mathcal{P}_H$ corresponding to the lowest "volatility matrix" $\underline{a}$. Furthermore, the robust good-deal hedging strategy $\bar{\phi}:=\bar{\phi}(X)$ for the put option $X=(\mathcal{K}-L_T)^+$ is given by $\widehat{a}_t^{1/2}\bar{\phi}_t=\widehat{\Pi}_t(\widehat{a}_t^{1/2}Z_t)$, for $Z=(Z^1,Z^2)^{\mathsf{tr}}$ given by (4.40), i.e.

$$\bar{\phi}_t=-\beta e^{m(T-t)}N(-d_+)L_t\,\widehat{a}_t^{-1/2}\,\widehat{\Pi}_t\Big(\widehat{a}_t^{1/2}\big(\rho,\sqrt{1-\rho^2}\big)^{\mathsf{tr}}\Big),\quad\text{for all } t\in[0,T],\ \mathcal{P}_H\text{-q.s..}$$

Now for a vector $z=(z_1,z_2)^{\mathsf{tr}}\in\mathbb{R}^2$, straightforward calculations using (4.36) and (4.35) imply

$$\begin{aligned}
\widehat{a}^{-1/2}\widehat{\Pi}(\widehat{a}^{1/2}z)&=\frac{1}{\widehat{b}^{11}\widehat{b}^{22}-(\widehat{b}^{12})^2}\begin{pmatrix}\widehat{b}^{22} & -\widehat{b}^{12}\\ -\widehat{b}^{12} & \widehat{b}^{11}\end{pmatrix}\\
&\quad\cdot\frac{1}{(\widehat{b}^{11})^2+(\widehat{b}^{12})^2}\begin{pmatrix}\big((\widehat{b}^{11})^3+\widehat{b}^{11}(\widehat{b}^{12})^2\big)z_1+\big(\widehat{b}^{12}(\widehat{b}^{11})^2+\widehat{b}^{11}\widehat{b}^{12}\widehat{b}^{22}\big)z_2\\ \big(\widehat{b}^{12}(\widehat{b}^{11})^2+(\widehat{b}^{12})^3\big)z_1+\big(\widehat{b}^{11}(\widehat{b}^{12})^2+\widehat{b}^{22}(\widehat{b}^{12})^2\big)z_2\end{pmatrix}\\
&=\frac{1}{\widehat{a}^{11}\big(\widehat{b}^{11}\widehat{b}^{22}-(\widehat{b}^{12})^2\big)}\begin{pmatrix}\big(\widehat{b}^{11}\widehat{b}^{22}-(\widehat{b}^{12})^2\big)(\widehat{a}^{11}z_1+\widehat{a}^{12})z_2\\ 0\end{pmatrix}\\
&=\begin{pmatrix}z_1+\frac{\widehat{a}^{12}}{\widehat{a}^{11}}z_2\\ 0\end{pmatrix}.
\end{aligned}$$

Hence an explicit formula for $\bar{\phi}_t$ is

$$\bar{\phi}_t=-\beta e^{m(T-t)}N(-d_+)L_t\Big(\rho+\frac{\widehat{a}_t^{12}}{\widehat{a}_t^{11}}\sqrt{1-\rho^2},0\Big)^{\mathsf{tr}},\ \text{for all } t\in[0,T],\ \mathcal{P}_H\text{-q.s..}\tag{4.45}$$

As the optimal growth rate bound h tends to infinity, the good-deal bound $\pi_\cdot^u(X)$ increases towards the robust upper no-arbitrage bound under volatility as studied in [ALP95, Lyo95, DM06, NS12, Vor14]. The put option $X=(\mathcal{K}-L_T)^+$ being a claim with convex payoff function, our result agrees with those of [ALP95, Lyo95, EJPS98, Vor14] according to which in the presence of volatility uncertainty, no-arbitrage valuation of put options under maximal (resp. minimal) volatility corresponds to the worst-case for the seller (resp. buyer). The latter works focus on the robust super-replication problem under volatility uncertainty for valuation with respect to the worst-case no-arbitrage bound. Here we instead study the robust good-deal hedging problem under volatility uncertainty for valuation with respect to the worst-case good-deal bound. Let us also mention that [ALP95, EJPS98, Vor14] work in a one-dimensional model with a single risky asset and obtain as super-replicating strategy the delta of the option under the worst-case measure. This is included in our case study as a special case for $|\rho| = 1$. In a generalization towards a two-dimensional model, we consider possibly non-perfectly correlated (traded and non-traded) risky assets and derive a robust good-deal hedging strategy for the worst-case good-deal valuation in a market that is possibly incomplete under each fixed prior $P \in \mathcal{P}_H$. Furthermore the robust good-deal hedging strategy $\bar{\phi}$ here is not (the risky asset component of) the super-replicating strategy, in particular, when $0 < |\rho| < 1$. Indeed since $0 \le X \le \mathcal{K}$ holds pathwise, the no-arbitrage bound process $\widehat{V}(X)$ under $P^{\overline{a}}$ defined by

$$\widehat{V}_t(X) := \underset{Q \in \mathcal{M}^e(P^{\overline{a}})}{\text{ess sup}^{P^{\overline{a}}}} E_t^Q[X],\ t \in [0,T],\ P^{\overline{a}}\text{-a.s.}$$

satisfies $\pi_t^{u,P^{\overline{a}}}(X;h) \le \widehat{V}_t(X) \le \mathcal{K},\ P^{\overline{a}}$-a.s. for $\pi_t^{u,P^{\overline{a}}}(X;h)$ given by $\pi_t^{u,P^{\overline{a}}}(X)$ in (4.39). In addition if $|\rho| < 1$ then $\pi_t^{u,P^{\overline{a}}}(X;h) \nearrow \mathcal{K}$ as $h \nearrow +\infty$ (since then $m \to -\infty$, $d_\pm \to -\infty$). These imply that if $|\rho| < 1$ then

$$\widehat{V}_t(X) = \mathcal{K}\mathbb{1}_{\{t<T\}} + X\mathbb{1}_{\{t=T\}},\ t \in [0,T],\ P^{\overline{a}}\text{-a.s..} \tag{4.46}$$

Now by [Kra96, Theorem 3.2] the process $\widehat{V}(X)$ has the optional decomposition

$$\widehat{V}_t(X) = \widehat{V}_0(X) + \int_0^t \widehat{\phi}_s dB_s^1 - \widehat{C}_t, \quad t \in [0,T],\ P^{\overline{a}}\text{-a.s.},$$

where $\int_0^\cdot \widehat{\phi}_s dB_s^1$ and $\widehat{C}$ are unique (see [Kra96, Theorem 2.1 and Lemma 2.1]). This implies from (4.46) that $\int_0^\cdot \widehat{\phi}_s dB_s^1 = 0$ and $C = (\mathcal{K} - X)\mathbb{1}_{\{T\}}$. Note from (4.45) that $\bar{\phi} = (Z^1, 0)^{\mathrm{tr}}$, $P^{\overline{a}} \otimes dt$-a.s. since $\widehat{a} = \overline{a}$, $P^{\overline{a}} \otimes dt$-a.s.. For $\rho \ne 0$, the process Z^1 is non-trivial under $P^{\overline{a}} \otimes dt$, and thus $\int_0^\cdot \bar{\phi}_s^{\mathrm{tr}} dB_s = \int_0^\cdot Z_s^1 dB_s^1$ cannot be equal to $\int_0^\cdot \widehat{\phi}_s dB_s^1 \equiv 0,\ P^{\overline{a}} \otimes dt$ almost surely if $0 < |\rho| < 1$. This means that for $0 < |\rho| < 1$ the good-deal hedging strategy $\bar{\phi}$ is different from the risky asset component of a super-replicating strategy for the model $P^{\overline{a}}$. Therefore, $\bar{\phi}$ is in general not the super-replicating strategy under volatility uncertainty for the set $\mathcal{P}_H$ of reference priors and $\pi_t^u(X) + \int_t^T \bar{\phi}_s^{\mathrm{tr}} dB_s$ does not dominate the claim X $P^{\overline{a}}$-almost surely, let alone $\mathcal{P}_H$-quasi-surely.

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List of Figures

List of Tables

Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe und ich zum ersten mal eine Doktorarbeit in diesem Studiengang einreiche.

Berlin, den 31. Juli 2015 Klébert Kentia Tonleu